

Post-Classical Probability Theory

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1 Introduction

This chapter offers a brief introduction to what is often called the *convex-operational approach* to the foundations of quantum mechanics, and reviews selected results, mostly by ourselves and collaborators, obtained using that approach. Broadly speaking, the goal of research in this vein is to locate quantum mechanics within a very much more general, but conceptually very straightforward, generalization of classical probability theory. The hope is that, by regarding QM from the outside, so to say, we shall be able to understand it more clearly. And, in fact, this proves to be the case.

The phrase “convex-operational” deserves some comment. The approach discussed here is “convex” in that it takes the space of states of a physical system to be a convex set (to accommodate the formation of probabilistic mixtures), and draws conclusions from the geometry of this set. It is “operational” in its acceptance of measurements and their outcomes as part of its the primitive conceptual apparatus, and in its identification of states with probability weights on measurement outcomes. In this sense, it is conceptually very conservative, differing from classical probability only in that it is *not* assumed that all measurements can be made simultaneously.

From this starting point, one is led very naturally to a mathematical framework for a *post-classical* probability theory, which, while varying idiomatically from author to author [9, 27, 29, 32, 39, 40, 46, 49], is more or less canonical. About the first third of what follows is devoted to a detailed discussion of the structure of individual probabilistic models in this framework. Here we exhibit a range of simple non-classical examples, many of them quite different from *either* classical or quantum probabilistic models. At the same time, we try to bring some order to this diversity, by showing that essentially any probabilistic model can be represented in a natural way in terms of an ordered real vector space and its dual, and that processes operating on and between models can be reresented by positive linear maps between these associated spaces.

Starting in Section 3, we focus on *composites* of probabilistic models, subject to a natural non-signaling constraint. As we shall see, the phenomenon of *entanglement*, often regarded as a hallmark of quantum mechanics, is actually a rather generic feature of non-signaling composites of non-classical state spaces, and thus, more a marker of non-classicality than of “quantumness” *per se*. Since quantum information theory treats entanglement as a resource, the question then arises of which quantum-information theoretic results can be made to work in a more general probabilistic setting. Section 4 reviews some work in this direction, particularly the generalization of the no-cloning and no-broadcasting theorems of [9, 10], and the analysis of teleportation and entanglement-swapping protocols in terms of conditional states, following [11].

If many non-classical features of QM are not so much quantum as generically non-classical, what *does* single out QM? The question of how to characterize QM in operational or probabilistic terms is a very old one. After many decades of hard-won partial results in this direction (e.g., [4, 5, 22, 37, 54, 63, 75]), the past decade has produced a slew of novel derivations of finite-dimensional QM from fairly simple, transparent and plausible, assumptions [23, 26, 39, 48, 55] (to cite just a few). In Section 5, we outline one of these, which recovers the Jordan structure of finite-dimensional

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quantum theory from symmetry considerations; the specific C^* -algebraic machinery of standard quantum mechanics is then singled out by considerations involving the formation of composite systems. The key tools here are a classical representation theorem for homogeneous, self-dual cones, due to M. Koecher and E. Vinberg [42, 65], and a theorem about tensor products of Jordan algebras due to H. Hanche Olsen [38].

Since the aim of this paper is to provide a brief and accessible introduction to this material, we make some simplifying assumptions. The most important is that we focus *entirely* on finite-dimensional models, even though large parts of the apparatus developed here work perfectly well (and were first developed) in an infinite-dimensional setting. Further assumptions will be spelled out as we go.

Notational conventions Real vector spaces are indicated generically by bold capitals \mathbf{E}, \mathbf{F} , etc. The space of linear mappings $\mathbf{E} \rightarrow \mathbf{F}$ is denoted by $L(\mathbf{E}, \mathbf{F})$; \mathbf{E}^* denotes the dual space of \mathbf{E} . If \mathcal{H} is a real or complex Hilbert space, $\mathcal{L}_h(\mathcal{H})$ stands for the space of bounded Hermitian operators on \mathcal{H} . If X is a set, \mathbb{R}^X denotes the vector space of all real-valued functions on X .

2 Elementary probability theory, classical and otherwise

If \mathcal{H} is a Hilbert space, representing a quantum-mechanical system, then each state of that system is represented by a density operator ρ . A possible measurement outcome is represented by an *effect*, i.e., a positive hermitian operator a with $\mathbf{0} \leq a \leq \mathbf{1}$; $\text{Tr}(\rho a)$ gives the *probability* that a will occur (if measured) when the state ρ obtains. This probabilistic apparatus generalizes that of classical probability theory, in that if we fix an *observable*, that is, a set $\{a_1, \dots, a_n\}$ of effects summing to $\mathbf{1}$, we can understand this as a model of a single, discrete, classical statistical experiment, on which each state ρ defines a probability weight $p(i) := \text{Tr}(\rho a_i)$. The novelty here is that, in general, a pair of observables $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_k\}$ is not *co-measurable*. In classical probability theory, it is always assumed (if often tacitly) that any pair of outcome-sets E_1 and E_2 admit a simultaneous refinement, that is, both can be represented as partitions or “coarse-grainings” of some third outcome-set F . In quantum-probability theory, this is not the case. Unless the operators a_i and b_j all commute, there will be no third observable of which E_i are both coarse-grainings.

So, quantum probability theory foregoes the assumption of co-measurability, which is a tenet of classical probability theory. And, indeed, in retrospect, the latter is surely a *contingent* matter, so it is not so very radical a step to renounce it. It is not so much the intuitive notion of probability that is post-classical, as the overall framework, which is in a precise sense a generalization of the framework of the classical mathematical theory of probability. On the other hand, quantum probability theory replaces the simple axiom of co-measurability with the elaborate apparatus of the Hilbert space \mathcal{H} and its associated space of Hermitian operators. As a framework for an autonomous probability calculus, this seems less than perfectly well motivated, and one can wonder whether, and why, it is necessary. A sensible way to approach this question is simply to drop the co-measurability assumption, without making any special assumptions to replace it. The resulting *post-classical* probability theory is a vast, poorly explored, and rather wild region, within which even quantum probability theory seems rather tame.

2.1 Test spaces and probabilistic models

There are many more or less equivalent, but stylistically diverse, ways of formulating a post-classical probability theory. The approach we take here (due originally to C. H. Randall and D. J. Foulis [31, 32]) begins with a very minimum of raw material.

Definition 1. A test space is a pair (X, \mathcal{M}) where X is a set of outcomes and \mathcal{M} is a covering of X by non-empty sets called tests. A probability weight on (X, \mathcal{M}) is a function $\alpha : X \rightarrow [0, 1]$ with $\sum_{x \in E} \alpha(x) = 1$ for every $E \in \mathcal{M}$.

The intended interpretation is that each $E \in \mathcal{M}$ is the set of mutually exclusive outcomes associated with some probabilistic experiment — anything from rolling a die to asking a question to

making a measurement (via some well-defined procedure) of some physical quantity. It is permitted that distinct tests may overlap, that is, that distinct experiments may share some outcomes. The definition of a probability weight requires that, when this is the case, the probability of a given outcome be independent of the measurement used to secure it. In other words, probability weights are *non-contextual*.¹

It will be convenient to use the same letter, X , to denote the entire test space (X, \mathbf{M}) , as well as its outcome-set, leaving the set of tests tacit. When necessary, we'll write $\mathbf{M}(X)$ for the latter. We also write $\Omega(X)$ for the set of all probability weights on X . This is a *convex* subset of $[0, 1]^X \subseteq \mathbb{R}^X$, i.e.,

$$\alpha, \beta \in \Omega(X) \Rightarrow t\alpha + (1-t)\beta \in \Omega(X)$$

for all $0 \leq t \leq 1$. Where X is *locally finite*, meaning that every test $E \in \mathbf{M}(X)$ is a finite set, it is not hard to see that $\Omega(X)$ is closed, and hence compact, with respect to the product topology on $[0, 1]^X$. It follows that $\Omega(X)$ is the closed convex hull of its extreme points.

Models In constructing a model for a probabilistic system, we may wish to single out certain probability weights as corresponding to possible *states* of the system. It is reasonable to form probability-weighted averages of such states, in order to represent ensembles of systems in different states. It is also reasonable to idealize the situation slightly by assuming that the limit of a sequence of possible states should again count as a possible state. In the same spirit, we shall assume in what follows that X carries a Hausdorff topology, with respect to which states are continuous. This is harmless, since we can always use the discrete topology as a default.² Indeed, given that our focus here is exclusively on finite-dimensional models, it is not unreasonable to assume that X is even compact.

To make all of this official:

Definition 2. A *probabilistic model* — or, for purposes of this paper, just a *model* — is a structure (X, Ω) , where X is a Hausdorff test space and Ω is a pointwise-closed (hence, compact), convex set of continuous probability weights on $\Omega(X)$. The extreme points of Ω are the *pure states* of the model.

Notation: We henceforth use capital letters A, B , etc. to denote models, writing, e.g., $(X(A), \mathbf{M}(A))$ for the test space belonging to model A , and $\Omega(A)$ for A 's state space. (So technically, $A = ((X(A), \mathbf{M}(A)), \Omega(A))$.)

Example 1 (Classical Models). (a) The simplest classical models have the structure $(E, \Delta(E))$, where E is a single test (so that $\mathbf{M}(E) = \{E\}$), and where $\Delta(E)$ is the simplex of all probability weights thereon. We might also deem “classical” a broader set of models: those of the form (E, Ω) where $\Omega \subseteq \Delta(E)$ is any closed, convex set of probability weights sufficiently large to statistically separate different outcomes³ of the single test E .

(b) A more sophisticated classical model begins with a measurable space S , and identifies statistical experiments with finite or countably infinite partitions of S by measurable subsets. The collection of all such experiments is a test space: let $X(S)$ be the set of non-empty measurable subsets of S (say, with the discrete topology), and let $\mathcal{D}(S)$ be the set of countable partitions of S into measurable subsets. We call $(X(S), \mathcal{D}(S))$ the *Kolmogorovian* test space associated with S . Probability weight on $(X(S), \mathcal{D}(S))$ correspond exactly to countably-additive probability measures on S .⁴

¹The formalism easily accommodates contextual probability assignments, however: simply define \tilde{X} to be the disjoint union of the test in \mathbf{M} — say, to be concrete, $\tilde{X} = \{(x, E) | x \in E \in \mathbf{M}\}$. In effect, each outcome of \tilde{X} consists of an outcome of X , plus a *record* of which test was used to secure it. For each test $E \in \mathbf{M}$, let $\tilde{E} = \{(x, E) | x \in E\}$, and let $\tilde{\mathbf{M}} = \{\tilde{E} | E \in \mathbf{M}\}$. Probability weights on $(\tilde{X}, \tilde{\mathbf{M}})$ are exactly what one means by *contextual* probability weights on (X, \mathbf{M}) . There is a natural surjection $\tilde{X} \rightarrow X$ that simply forgets these records; probability weights on (X, \mathbf{M}) pull back along this surjection to give us weights on $(\tilde{X}, \tilde{\mathbf{M}})$.

²A more detailed discussion of test spaces with topological structure can be found in [68]

³That is, given any pair of distinct outcomes, there exists a state assigning them different probabilities.

⁴By varying $\mathcal{D}(S)$, we can change the character of the probability weights that are allowed. For example, if we let $\mathcal{D}(S)$ include just the *finite* measurable partitions of S , then probability weights on $\mathcal{D}(S)$ correspond to finitely additive measures on S .

Example 2 (Quantum Models). (a) The most basic quantum-mechanical model begins with a complex Hilbert space \mathcal{H} . The *quantum test space* is $(X(\mathcal{H}), \mathbf{M}(\mathcal{H}))$ where the outcome space $X(\mathcal{H})$ is the unit sphere of \mathcal{H} (with its usual topology) and where the space $\mathbf{M}(\mathcal{H})$ of tests is the set of unordered orthonormal bases of *frames* of \mathcal{H} . Every unit vector $v \in \mathcal{H}$ determines a probability weight α_v on $\mathbf{M}(\mathcal{H})$, defined for all $x \in X(\mathcal{H})$ by

$$\alpha_v(x) = |\langle v, x \rangle|^2 = \text{Tr}(P_v P_x),$$

where P_v and P_x are the rank-one projection operators corresponding to v and x . Accordingly, if W is a density operator on \mathcal{H} — a positive hermitian operator of trace one, or, equivalently, a convex combination of rank-one projections — then $\alpha_W(x) := \langle Wx, x \rangle = \text{Tr}(WP_x)$ defines a probability weight on $X(\mathcal{H})$. If $\dim(\mathcal{H}) \geq 3$, then Gleason's theorem tells us that every probability weight on $X(\mathcal{H})$ is of this form, but for $\dim(\mathcal{H}) = 2$, there are many others, which one regards as non-physical. In either case, letting $\Omega(\mathcal{H})$ denote the convex set of density operators on \mathcal{H} , we obtain the *quantum model* $A(\mathcal{H}) = (X(\mathcal{H}), \Omega(\mathcal{H}))$.

A slightly different model, which we'll call the *projective quantum model*, and which we denote by $A(\mathbb{P}\mathcal{H})$, replaces each outcome $x \in X(\mathbb{P}\mathcal{H})$ by the corresponding rank-one projection operator P_x ; tests in $\mathbf{M}(\mathbb{P}\mathcal{H})$ are maximal pairwise orthogonal families of such projections. Again, states correspond to density operators via the recipe $\alpha_W(P_x) = \text{Tr}(WP_x)$ where $P_x \in X(\mathbb{P}\mathcal{H})$. For many purposes, the choice between $A(\mathcal{H})$ and $A(\mathbb{P}\mathcal{H})$ is one of convenience. However, notice that in passing from $A(\mathcal{H})$ to $A(\mathbb{P}\mathcal{H})$ we lose information about phase relations between the unit vectors representing outcomes of $X(\mathcal{H})$, which are important in describing sequential experiments. We won't pursue this here. The paper [74] contains some relevant discussion.

(b) A more sophisticated quantum model might begin with a W^* -algebra \mathcal{A} , and take for \mathbf{M} , the collection of all (say, finite) sets of projections summing to the identity in \mathcal{A} . If \mathcal{M} has no I_2 summand, the Christensen-Yeadon extension of Gleason's theorem [28] identifies the probability weights on \mathbf{M} with states on \mathcal{A} . Again, if there are I_2 factors (copies of $M_2(\mathbb{C})$), then one must explicitly limit the states to the quantum-mechanical ones.

By the *dimension* of a model A , we mean the dimension of the span of $\Omega(A)$ in $\mathbb{R}^{X(A)}$. Of course, this will generally be infinite. However, as mentioned in the introduction, our focus in this paper is on finite-dimensional models. Indeed, making this official, we assume from this point forward that **all models are finite-dimensional**. In particular, all *quantum models* $A(\mathcal{H})$ and $A(\mathbb{P}\mathcal{H})$ involve only finite-dimensional Hilbert spaces \mathcal{H} .

If we let $\mathbf{V}(\Omega)$ denote the span of Ω in $\mathbb{R}^{X(A)}$, we can map $X(A)$ into $\mathbf{V}(A)^*$ by evaluation. That is, for each outcome $x \in X(A)$, there is a canonical evaluation functional $\hat{x} : \mathbf{V}(A) \rightarrow \mathbb{R}$ given by $\hat{x}(\alpha) = \alpha(x)$. It may happen that, for some sequence x_i of outcomes, $\hat{x}_i \rightarrow a \in \mathbf{V}(A)^*$. Let us say that A is *outcome-closed* iff every such limit again corresponds to an outcome in $X(A)$, i.e. that there exists some $x \in X(A)$ with $a = \hat{x}$. Where $X(A)$ is compact in its native topology — which, in finite dimensional examples, it very often is — this condition is automatically satisfied. We make it another standing assumption that **all models are outcome-closed**.

Dispersion-Free States and Distinguishability One very striking difference between classical and quantum models has to do with the existence of (globally) *dispersion-free*, that is, zero-or-one valued, states. In both of the classical models considered above, all pure states are dispersion-free. Quantum models, in contrast, have *no* dispersion-free state: a pure quantum state still makes only uncertain predictions about the results of most measurements.

Definition 3. A set Ω of probability weights on a test space X is *unital* iff, for every $x \in X$, there exists at least one $\alpha \in \Omega$ with $\alpha(x) = 1$. If there is a *unique* such state, we say that Ω is *sharp*. We say that a model A is unital or sharp if its state space $\Omega(A)$ is a unital, respectively sharp, set of probability weights on the test space $X(A)$.

Like the classical examples, the quantum models $A(\mathcal{H})$ and $A(\mathbb{P}\mathcal{H})$ are sharp; indeed, the unique state α assigning probability one to a given outcome $x \in X(\mathcal{H})$, or to the corresponding outcome $P_x \in X(\mathbb{P}\mathcal{H})$, is the one corresponding to the density operator P_x .

Definition 4. A set Ω of probability weights on a test space X *separates outcomes*, or is *separating*, iff, for all outcomes $x, y \in X$, $\alpha(x) = \alpha(y)$ for all $\alpha \in \Omega$ implies $x = y$. A model A is *separated* iff $\Omega(A)$ separates outcomes of $X(A)$.

The state space of a standard quantum model $A(\mathfrak{H})$ is not separating; that of the corresponding projective quantum model $A(\mathbb{P}\mathfrak{H})$ is separating. As this example illustrates, given a non-separated model A , one *can* always replace $X(A)$ by an obvious quotient test space, in which probabilistically indistinguishable outcomes are identified, to obtain a separated model having the same states. One may or may not *wish* to do so.

A *partition space* is a test space that is isomorphic⁵ to a sub-test space of $\mathcal{D}(S)$ for some set S . Any such space supports a state-separating set of dispersion-free probability weights, namely, the point-masses associated with the points of S . The following is straightforward:

Lemma 1. *If test space has a unital, separating set of dispersion-free states, then it is a partition test space. If it has a sharp set of unital, DF states, then it is classical.*

In anticipation of later results, we'll write $x \perp y$ to mean that outcomes $x, y \in X(A)$ are *distinguishable* by means of some test $E \in X(\mathfrak{M})$ — that is, that $x, y \in E$ and $x \neq y$. At present, there is no linear structure in view, let alone an inner product, so the notation is only suggestive. Later, we'll see that one can often embed X in an inner product space in such a way that the notation can be taken literally.

It will also be useful to introduce the following notion of distinguishability for *states*.

Definition 5. Two states, $\alpha, \beta \in \Omega(A)$ are *sharply distinguishable* iff there exist outcomes $x, y \in X(A)$ with $x \perp y$ such that $\alpha(x) = \beta(y) = 1$. More generally, states $\alpha_1, \dots, \alpha_n$ are *jointly* sharply distinguishable iff there exists a test $E \in \mathfrak{M}(A)$ and outcomes $x_1, \dots, x_n \in E$ with $\alpha_i(x_j) = \delta_{i,j}$.

The idea is that, if the system is known to be in one of the states $\alpha_1, \dots, \alpha_n$, then by performing the measurement E we will learn — with probability one — which of these states was the actual one.⁶

2.2 Further Examples

Classical and quantum examples hardly exhaust the possibilities, of course: the whole point of the present framework is to provide us with a maximum of flexibility in constructing *ad hoc* models.

Example 3 (The Square Bit). The very simplest non-classical model starts with a test space X be a test space containing just two tests $E = \{x, x'\}$ and $F = \{y, y'\}$, each having two outcomes — as, say, two coins, or a stern-Gerlach apparatus with two angular settings. The convex set $\Omega(X)$ of all probability weights on X is affinely isomorphic to the unit square, under the mapping $\alpha \mapsto (\alpha(x), \alpha(y))$. The model (X, Ω) has, accordingly, been called the *square bit* [12]. As $\Omega(X)$ is not a simplex, this model is not entirely classical. On the other hand, as its pure states are all dispersion-free, it is very far from being “quantum”.

Greechie Diagrams A useful graphical device for representing small test spaces (those involving only a few outcomes) is to represent each outcome as a dot, and to join outcomes belonging to a test by a straight line or other smooth arc, with arcs corresponding to distinct tests intersecting, if at all, at a sharp angle, so as to be easily distinguished. Such a representation (first used in the quantum-logical literature) is called a *Greechie diagram* [36]. For example, we might represent a three-outcome classical test by the diagram in Figure 2 (a), and the square-bit test space by that in Figure 2 (b). The test space pictured in (c), with two three-outcome tests (the top and bottom

⁵An isomorphism of test spaces is a bijection from outcomes to outcomes, preserving tests in both directions.

⁶A weaker notion would require only that $\alpha_i(x_i) > 0 = \alpha_i(x_j)$ for each i, j , so that with *some* non-zero probability we obtain either x or y , and thus learn which state was actual. Notice, too, that the condition of joint sharp distinguishability is a priori much stronger than pairwise sharp distinguishability.

rows) and three two-outcome tests (the vertical lines), makes the point that a test space need not have any states at all.

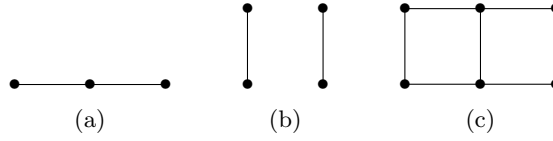


FIGURE 1: VARIOUS GREECHIE DIAGRAMS

The following whimsical example (due to D. J. Foulis) is useful as an antidote to several too-comfortable intuitions.

Example 4 (The Firefly Box). Suppose a sealed triangular box is divided into three interior chambers, as in the top-down view in Figure 2(a), below. The walls of the box are translucent, while the top, the bottom, and the interior partitions are opaque. In the box is a firefly, free to move about between the chambers (for which purpose, the interior partitions contain small tunnels). Viewed from one side, we might see the firefly flashing in chamber a or chamber b , or we might see nothing – the firefly might not be flashing, or might be in chamber c . Thus, we have three experiments, corresponding to the three walls of the box: $\{a, x, b\}$, $\{b, y, c\}$ and $\{c, z, a\}$, where x, y and z are the (distinct) “no-light” outcomes associated with each experiment. The resulting test space $\mathfrak{A} = \{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}\}$ has the Greechie diagram pictured in Figure 2(b) below.

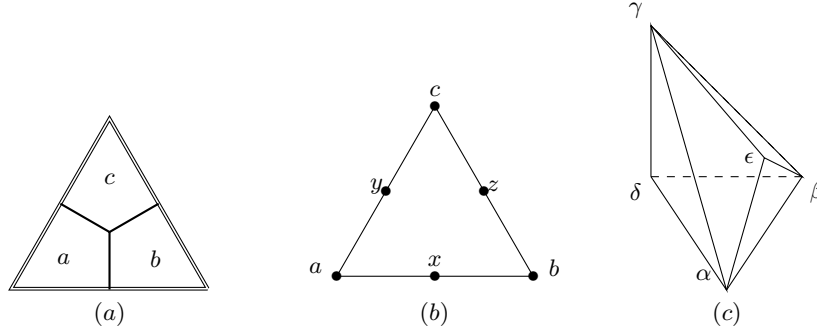


FIGURE 2: THE FIREFLY BOX

We can identify several pure states on this test space with concrete situations involving the location, and the internal state (lit or unlit) of the firefly. For example,

$$\alpha(a) = \alpha(z) = 1; \alpha(b) = \alpha(c) = \alpha(x) = \alpha(y) = 0$$

corresponds to the firefly’s flashing in chamber a . We can define similar states β and γ corresponding to chambers b and c . All of these states are dispersion-free. A fourth dispersion-free pure state, δ , assigns probability 1 to the outcomes x, y and z . This corresponds to the firefly not flashing. These four dispersion-free states separate outcomes separate the six outcomes, and thus allow us, by Lemma 1, to represent the firefly box as a partition test space over a classical state space. *However*, there is also a fifth, *non*-dispersion free pure state, ϵ , given by

$$\epsilon(a) = \epsilon(b) = \epsilon(c) = 1/2; \epsilon(x) = \epsilon(y) = \epsilon(z) = 0.$$

This last state is difficult to interpret in any way but to imagine that the firefly *responds* to being observed through a given window by entering (with equal probability) one of the two corresponding chambers. Since any state on this test space is determined by its values at the outcomes x, y and z , the convex set of all probability weights for the firefly box is a non-simplicial set in \mathbb{R}^3 : the pure states α, β and γ correspond to the standard basis vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, δ corresponds to the origin, and ϵ , to the vector $1/2(1, 1, 1)$. Thus, Ω is affinely isomorphic to a triangular diprism, as pictured in Figure 2 (c).

Example 5 (Grids and Graphs). Let E be a finite set — for definiteness, say $\{0, 1, \dots, n-1\}$, with $n \geq 2$. We define two test spaces associated with E :

- (a) The *grid test space*, $\mathbf{Gri}(E)$, consists of all rows and columns of the $n \times n$ array $E \times E$, that is, all sets of the form $\{x\} \times E$ or $E \times \{y\}$.
- (b) The *graph test space*, $\mathbf{Gra}(E)$ consists of the *graphs* of permutations $f : E \rightarrow E$, that is, subsets of $E \times E$ of the form $\{(i, f(i)) | i \in E\}$.

Both of these test spaces have outcome-set $X = E \times E$, so a state on either test space can be regarded as an $n \times n$ real matrix with non-negative entries. In the case of $\mathbf{Gri}(E)$, these entries must sum to unity along each row and column; that is, the states on $\mathbf{Gri}(E)$ are exactly the *doubly stochastic matrices*. By the Birkhoff-von Neumann theorem, these all arise as convex combinations of permutation matrices — that is, of the dispersion-free states corresponding to elements of $\mathbf{Gra}(E)$. Similarly, one can show that, for $n \geq 3$, every state of $\mathbf{Gra}(E)$ is an average of *row states*, α^k , given by $\alpha^k(i, j) = \delta_{i,k}$ and *column states* α_k , given by $\alpha_k(i, j) = \delta_{k,j}$.

Every pair of pure states on either $\mathbf{Gri}(E)$ or $\mathbf{Gra}(E)$ is distinguishable by a test in that space. Nevertheless, neither state space is a simplex for $n \geq 3$. The space of doubly-stochastic matrices has $n!$ pure states, which, for $n \geq 4$, exceeds the $n^2 + 1$ states permissible for a simplex in \mathbb{R}^{n^2} . For $n \geq 3$, $\mathbf{G}(E)$ has only $2n$ pure states; however, the maximally mixed state $\alpha(i, j) \equiv 1/n$, can be represented as a uniform average over just the row states, or over just the column states; similarly, on $\mathbf{Gri}(E)$, it can be represented as a uniform average over any set of permutations the graphs of which partition $E \times E$. By a curious coincidence, the test spaces $\mathbf{Gri}(3)$ and $\mathbf{Gra}(3)$ are isomorphic, so the state space of $\mathbf{Gri}(3)$ is isomorphic to that of $\mathbf{Gra}(3)$, and again, not a simplex.

Remark: We’ve seen that a variety of convex geometries can arise more or less naturally as the (full) state spaces of test spaces. A natural question is whether *every* possible convex geometry arises in this way. A theorem of F. Shultz [57] shows that in fact, every compact convex set can be represented as the space of probability measures on an orthomodular lattice. The set of decompositions of the unit element in such a lattice is a test space, the probability weights on which correspond precisely to the probability measures on the lattice. Thus, Shultz’ theorem implies that every compact convex set can be realized as the full state space of a test space.

Models from Symmetry A *symmetry* of a test space X is a bijection $g : X \rightarrow X$ such that both g and g^{-1} preserve tests — in other words, such that for all $E \subseteq X$, we have $gE \in \mathbf{M}(X)$ iff $E \in \mathbf{M}(X)$. (In other words, it is an isomorphism from the test space X to itself.) The set of all symmetries of X is evidently a group, which we’ll denote by $G(X)$. There is a natural dual action of $G(X)$ on probability weights on X , given by $g\alpha := \alpha \circ g^{-1}$; a symmetry of a model $A = (X, \Omega)$ is a symmetry of A that also preserves Ω . Again, the symmetries of a model form a group, $G(A) \leq G(X(A))$.

Both classical and quantum test spaces are marked by very strong symmetry properties. In particular, the symmetry group of either kind of system acts transitively on pure states, and also on the set of tests; moreover, any permutation of the outcomes of any given test can be implemented by a symmetry of the entire system. (This is more or less trivial in the case of a classical system; for a quantum system, it amounts to the observation that any permutation of an orthonormal basis for a Hilbert space \mathcal{H} extends to a unitary operator on \mathcal{H} .) In contrast, no symmetry of the “firefly box” test space of Example 4 will exchange one of the outcomes a, b, c with one of x, y, z , since each of the former belongs to two tests, while each of the latter belongs only to one.

Definition 6. Let G be a group acting by symmetries on a test space X . We say X is *symmetric* under G , or *G-symmetric*, iff G acts transitively on $\mathbf{M}(X)$, and the stabilizer G_E of a test $E \in \mathbf{M}(A)$ acts transitively on E . If X is G -symmetric and G_E acts doubly transitively on E , then X is *2-symmetric* under G . If G_E acts as the full permutation group of E , we say that X is *fully G-symmetric*.

In fact, test spaces with these symmetry properties can be constructed very naturally [72]. Suppose one has a simple measuring device, which can be applied to a system of some sort to

produce outcomes in a set E . One might be able to apply this device *in different ways* — for example, by changing the orientation of the apparatus with respect to the system, or by adjusting some controllable physical parameters associated with the system. This suggests that we might be able to build a larger family of experiments — a test space, in other words — starting with the basic measurement E , and adding parameters that keep track of the various ways in which we might deploy it. In many cases, there will be a group G of “physical symmetries” acting on these parameters, and we can often reconstruct the desired test space simply from a knowledge of this group and its relationship to the test E . Specifically, there will be some subgroup H of G that acts to permute the outcomes of E . Let us suppose that H acts transitively on E , so that, for any reference outcome $x_o \in E$, every other outcome $x \in E$ has the form hx_o for some $h \in H$. If we let K be any subgroup of G such that $K \cap H = H_{x_o}$, where H_{x_o} is the stabilizer in H of a chosen reference outcome $x_o \in E$, and set $X = G/K$. Then there is a well-defined canonical H -equivariant injection $j : E \rightarrow X$ given by $j(x) = hK$ where $x = hx_o$. Let us identify E with its image under j , so that $E \subseteq X$, let \mathcal{G} be the orbit of E under G , i.e.,

$$\mathcal{G} := \{gE | g \in G\}.$$

The test space (X, \mathcal{G}) will automatically be symmetric, and will be 2-symmetric or fully symmetric under G as H acts doubly or fully transitively on E . We obtain a G -symmetric *model* by choosing any G -invariant, closed, convex set of probability weights on X .

The choice of the group K extending the stabilizer H_o has a large effect on the combinatorial structure of (X, \mathcal{G}) . For example, if $K = H_o$, then \mathcal{M} is a semi-classical test space consisting of disjoint copies of E ; in general, a larger choice of K will enforce non-trivial intersections among the tests gE with $g \in G$.

Example 6. As an illustration of this construction, let $E = \{0, 1, \dots, n-1\}$, and let U be the group of all unitary $n \times n$ matrices, acting in the usual way on $\mathcal{H} = \mathbb{C}^E$. Let $H \leq U$ be the subgroup consisting of permutation matrices, and K , the group of unitaries fixing e_0 , the column vector corresponding to $0 \in E$. Then $K \cap H$ is exactly the set of permutation matrices corresponding to permutations fixing 0, i.e., $K \cap H = H_0$. Now $X = G/K$ is the (projective) unit sphere of \mathcal{H} , and \mathcal{M} is the set of (projective) frames of \mathcal{H} . For another example, let H be the full permutation group $S(E)$ of E and set $G = S(E) \times S(E)$. Embedding H in G by $h \mapsto (h, e)$, the construction above produces the “grid” test space $\mathcal{Gri}(E)$ of Example 6. Using instead the diagonal embedding $h \mapsto (h, h)$ yields the “graph” test space $\mathcal{Gra}(E)$.

2.3 Models Linearized

In many situations, the outcomes of a test space are naturally represented as elements of a vector space. This is obviously the case for the quantum-mechanical examples discussed above, where outcomes are directly identified with unit vectors in \mathcal{H} or with rank-one projections in $L(\mathcal{H})$. One can also formulate classical probability theory in this way, by considering the space of random variables associated with a given measurable space, and identifying measurement outcomes (that is, measurable sets) with the corresponding indicator random variables.

In fact, subject to some fairly mild restrictions, such a representation is always available. The idea will be to construct, for each such a model $A = (X, \Omega)$, a real vector space $\mathbf{E}(A)$, and an embedding of $X \rightarrow \mathbf{E}(A)$, in such a way that states in Ω extend uniquely to linear functionals on $\mathbf{E}(A)$. In fact, $\mathbf{E}(A)$ will be an *ordered* real vector space, so we pause briefly to review this notion (for further details, see [2]).

Ordered Linear Spaces By a *cone* in a real vector space \mathbf{E} , we mean a convex subset closed under multiplication by non-negative scalars, and satisfying $K \cap -K = \{0\}$. K is *generating* iff it spans \mathbf{E} . It spans \mathbf{E} . An *ordered linear space* is a real vector space \mathbf{E} , equipped with a closed, generating cone \mathbf{E}_+ . Such a cone determines a (partial) ordering, invariant under translation and under positive scalar multiplication, on \mathbf{E} , namely $a \leq b$ iff $b - a \in \mathbf{E}_+$.⁷ Noticing that $a \geq 0$ iff $a \in \mathbf{E}_+$, we refer

⁷Some authors define ordered linear spaces without requiring that the positive cone be generating. For our purposes, the present definition is more useful.

to \mathbf{E}_+ as the *positive cone* of \mathbf{E} .

The basic example is the space \mathbb{R}^X of all real-valued functions on a set X , ordered pointwise. Thus,

$$(\mathbb{R}^X)_+ = \{f \in \mathbb{R}^X \mid f(x) \geq 0 \ \forall x \in X\}.$$

Another example, central to our concerns here, is the space $\mathcal{L}_h(\mathcal{H})$ of bounded *hermitian* operators on a Hilbert space \mathcal{H} (over either \mathbb{R} or \mathbb{C}). This space has a standard ordering, induced by the cone $\mathbf{L}_+(\mathcal{H})$ of positive semi-definite operators — that is, $a \in \mathbf{L}_+(\mathcal{H})$ iff $\langle ax, x \rangle \geq 0$ for all vectors $x \in \mathcal{H}$. More generally, the real vector space of self-adjoint elements of a C^* -algebra \mathcal{A} is ordered by the cone of elements of the form aa^* , $a \in \mathcal{A}$.

If \mathbf{E} and \mathbf{F} are ordered linear spaces, a linear mapping $f : \mathbf{E} \rightarrow \mathbf{F}$ is *positive* iff $f(\mathbf{E}_+) \subseteq \mathbf{F}_+$, i.e., $f(a) \geq 0$ whenever $a \geq 0$. An *order-isomorphism* between \mathbf{E} and \mathbf{F} is a positive, invertible linear mapping having a positive inverse. We'll denote the set of positive linear mappings $\mathbf{E} \rightarrow \mathbf{F}$ by $\mathbf{L}(\mathbf{E}, \mathbf{F})$. This is a cone in the space $\mathbf{L}(\mathbf{E}, \mathbf{F})$. As a special case, the dual space of an ordered vector space \mathbf{E} has a natural *dual cone*, $\mathbf{E}_+^* = \mathbf{L}_+(\mathbf{E}, \mathbb{R})$. In our present finite-dimensional setting, this is generating, so \mathbf{E}^* becomes an ordered vector space in a natural way.

Order-unit spaces An *order unit* in an ordered linear space \mathbf{E} is an element $u \in \mathbf{E}_+$ such that, for every $a \in \mathbf{E}$, there exists some $n \in \mathbb{N}$ with $a \leq nu$. When \mathbf{E} is finite-dimensional, this is equivalent to asking that $\alpha(u) > 0$ for every non-zero $\alpha \in \mathbf{E}_+^*$, which can always be arranged. (In particular, a finite-dimensional ordered linear space always *has* an order-unit.) An *order-unit space* is an ordered linear space equipped with a distinguished order-unit. The key example to bear in mind is the space $\mathcal{L}_h(\mathcal{H})$, ordered as described above, and with the identity operator as order-unit.

An order unit space already provides enough structure to support probabilistic ideas. A *state* on an order-unit space \mathbf{E} is a linear functional $\alpha \in \mathbf{E}^*$ with $\alpha(u) = 1$. An *effect* in \mathbf{E} is a positive element a with $a \leq u$, so that $0 \leq \alpha(a) \leq 1$ for every state α . A discrete *observable* on \mathbf{E} is a finite set $E = \{a_1, \dots, a_k\}$ of non-zero effects with $a_1 + \dots + a_k = u$; evidently, any state on \mathbf{E} restricts to a probability weight on every observable on \mathbf{E} . Thus, the observables form a test space, the outcomes of which are just the non-zero effects in \mathbf{E}_+ . In the special case where $\mathbf{E} = \mathcal{L}_h(\mathcal{H})$, the space of Hermitian operators on a Hilbert space \mathcal{H} , an effect is a positive operator a with $0 \leq a \leq \mathbf{1}$; all states have the form $\alpha(a) = \text{Tr}(Wa)$ where W is a density operator on \mathcal{H} , and an observable is essentially a (discrete) positive-operator valued measure.

The set of all (normalized) states on an order-unit space \mathbf{E} is the latter's *state space*. This is always a compact convex set. Conversely, if Ω is any compact convex subset of any finite-dimensional real vector space, let $\text{Aff}(\Omega)$ denote the space of bounded affine (that is, convex-combination preserving) real-valued functionals $f : \Omega \rightarrow \mathbb{R}$, ordered pointwise. The constant functional $u(\alpha) \equiv 1$ serves as an order unit. One can show that Ω (embedded in $\text{Aff}(\Omega)^*$ by evaluation) is exactly $\text{Aff}(\Omega)$'s state space. Moreover, if $T : \Omega \rightarrow \mathbf{W}_+$ is any affine mapping of Ω into the positive cone of a (finite-dimensional) ordered linear space \mathbf{W} , then T extends uniquely to a positive linear mapping $T : \mathbf{E}(\Omega)^* \rightarrow \mathbf{W}$.

The linear hull of a model Any probabilistic model can be interpreted, in a canonical way, in terms of an order-unit space with a distinguished family of observables. Let $A = (X, \Omega)$ be a probabilistic model. Every outcome $x \in X(A)$ determines an affine functional $\hat{x} : \Omega \rightarrow \mathbb{R}$ by evaluation: $\hat{x}(\alpha) = \alpha(x)$ for all $\alpha \in \Omega$.

Definition 7. If $A = (X, \Omega)$ is a model, write $\mathbf{E}(A)$ for the span of X in \mathbb{R}^Ω , ordered by the closure of the cone consisting of linear combinations with non-negative coefficients of evaluation functionals \hat{x} , $x \in X(A)$:

$$\mathbf{E}(A)_+ = \text{cl} \left(\left\{ \sum_i t_i \hat{x}_i \mid x_i \in X, t_i \geq 0 \right\} \right).$$

Letting $u \in \mathbb{R}^\Omega$ denote the constant function $u(\alpha) \equiv 1$, we see that $\sum_{x \in E} \hat{x}$, where E is any test in $\mathcal{M}(A)$. Hence, u belongs to \mathbf{E}_+ , where it functions as an order-unit. The order-unit space $(\mathbf{E}(A), u)$, together with the embedding $X(A) \rightarrow \mathbf{E}(A)$, is called the *linear hull* of the model A .

Every test $E \in \mathcal{M}(A)$ can now be regarded as a discrete observable on $\mathbf{E}(A)$. Notice that the cone $\mathbf{E}(A)_+$ may well be smaller than the cone $\{a \in \mathbf{E} \mid a(\alpha) \geq 0 \ \forall \alpha \in \Omega\}$ inherited from $\text{Aff}(\Omega(A))_+$, and that, unlike the latter, it depends on the choice of $X(A)$.

Example 7. In the case of a quantum model $A = (X(\mathcal{H}), \mathcal{M}(\mathcal{H}))$ of Example 2, the space $\mathbf{E}(A)$ — or, as we'll denote it below, $\mathbf{E}(\mathcal{H})$ — can be identified with the order-unit space $\mathcal{L}_h(\mathcal{H})$ of Hermitian operators on \mathcal{H} , ordered by the usual cone, with u the identity operator.

There is a canonical embedding of $\Omega(A)$ in $\text{Aff}(\Omega)^*$, taking each state $\alpha \in \Omega(A)$ with the corresponding evaluation functional $f \mapsto f(\alpha)$, $f \in \text{Aff}(\Omega)$. Let $\mathbf{V}(A)$ denote the span of $\Omega(A)$ in $\text{Aff}(\Omega)^*$, ordered by the cone $\mathbf{V}_+(A)$ generated by $\Omega(A)$. Since $\mathbf{E}(A) \leq \text{Aff}(\Omega)$, we have a natural duality between $\mathbf{V}(A)$ and $\mathbf{E}(A)$, or, to put it another way, there is a natural linear mapping $\mathbf{V}(A) \rightarrow \mathbf{E}(A)^*$, taking each $\alpha \in \Omega$ to the corresponding evaluation functional in $\mathbf{E}(A)^*$. Since states are, for us, probability weights on $X(A)$, this mapping is injective.

State-Completeness If $A = (X, \Omega)$ is a model, with linear hull $\mathbf{E}(A)$, then any positive linear functional $\alpha \in \mathbf{E}(A)^*$ with $\alpha(u) = 1$ (that is, any state *on* \mathbf{E}) defines a probability weight on $X(A)$ by restriction. Let $\widehat{\Omega}$ denote the set of such states. Obviously, $\Omega \subseteq \widehat{\Omega}$. We may regard $\widehat{\Omega}$ as the set of probability weights that are consistent with all of the linear relations among outcomes that are satisfied by the given state space Ω . Evidently, the assignment $\Omega \mapsto \widehat{\Omega}$ is a closure on the poset of closed convex subsets of $\Omega(X)$. Call a model *state-complete* iff $\Omega = \widehat{\Omega}$.

Lemma 2. *Let $A = (X, \Omega)$ be a finite-dimensional probabilistic model. Then the following are equivalent:*

- (a) *A is state-complete*
- (b) $\mathbf{E}(A)_+ = \mathbf{E}(A) \cap \text{Aff}_+(\Omega) = \mathbf{E}(A) \cap \mathbf{V}(A)^*$;
- (c) *The canonical mapping $\mathbf{V}(A) \rightarrow \mathbf{E}(A)^*$ is surjective, hence, an order-isomorphism.*

Proof: To see that (a) implies (b), suppose $f \in \text{Aff}_+(\Omega) \setminus \mathbf{E}(A)_+$. Then (by the finite-dimensional version of the Hahn-Banach separation theorem) there exists some $\alpha \in \mathbf{E}(A)^*$ with $\alpha(a) \geq 0$ for all $a \in \mathbf{E}(A)_+$ but $\alpha(f) < 0$. We can normalize α so that $\alpha(u) = 1$, in which case $\alpha \in \widehat{\Omega}$. Since f is non-negative on Ω , it follows that $\alpha \notin \Omega$, whence, $\widehat{\Omega} \neq \Omega$, and A is not state-complete. Conversely, if $\alpha \in \widehat{\Omega} \setminus \Omega$, then we can find some $f \in \mathbf{E}(A)^{**} = \mathbf{E}(A)$ with $f(\alpha) < 0$ but $f(\beta) \geq 0$ for all $\beta \in \Omega$. But now $f \in \mathbf{E} \cap \text{Aff}_+(\Omega)$, and yet — as $a(\alpha) \geq 0$ for all $a \in \mathbf{E}(A)_+$ — we have $f \notin \mathbf{E}(A)_+$. Thus, (b) implies (c). As all systems here are finite-dimensional, (b) and (c) are clearly equivalent. \square

Standing Assumption: *Henceforth, all models are state-complete.*

One might almost, at this point, regard the test space $X(A)$ as merely a sort of builder's scaffolding, to be discarded once the space $\mathbf{E}(A)$ has been constructed. For many applications, this works perfectly well. However, the additional structure represented by X turns out to be useful in many ways, so we prefer to retain it for present purposes. Doing so imposes no additional restrictions on the structure of $\mathbf{E}(A)$ because, given an order-unit space \mathbf{E} , we can always take X to consist of *all* observables on \mathbf{E} , as discussed above.⁸

Direct Sums of Models A *face* of a convex set K is a convex subset $J \subseteq K$ such that, for all $a, b \in K$ and all $0 \leq t \leq 1$,

$$ta + (1-t)b \in J \Rightarrow a \in K \text{ and } b \in K.$$

⁸One of many uses for the test space structure is to privilege certain classes of observables on an order-unit space having special order-theoretic properties — for example, the set of observables the outcomes of which lie on extremal rays of \mathbf{E}_+ forms a test space, or those whose outcomes are atomic effects, i.e., those that lie on extremal rays of \mathbf{E}_+ and are extreme points of $[0, u]$.

If J and K are cones, then this is equivalent to the condition that $a + b \in J \Rightarrow a \in J$ and $b \in J$. A minimal face of a cone is in fact a ray; we more usually speak of an *extremal ray*. An element of a cone is *ray-extremal*, or simply *extremal*, iff it generates an extremal ray. In finite dimensions, every (closed) cone is the convex hull of its extremal elements.

The *direct sum* of two ordered vector spaces \mathbf{E} and \mathbf{F} is their vector-space direct sum, $\mathbf{E} \oplus \mathbf{F}$, equipped with the cone $\mathbf{E}_+ \oplus \mathbf{F}_+$ consisting of all sums of positive elements from each. This is the smallest cone in $\mathbf{E} \oplus \mathbf{F}$ making the standard embeddings $\mathbf{E}, \mathbf{F} \rightarrow \mathbf{E} \oplus \mathbf{F}$ given by $a \mapsto (a, 0)$ and $b \mapsto (0, b)$ (for $a \in \mathbf{E}$ and $b \in \mathbf{F}$) positive. In this case, \mathbf{E}_+ and \mathbf{F}_+ are both faces of $\mathbf{E}_+ \oplus \mathbf{F}_+$. .. \mathbf{E} is irreducible iff not a direct sum.

If X and Y are sets, we write $X \oplus Y$ for their coproduct (or disjointified union),

$$X \oplus Y = \{1\} \times X \cup \{2\} \times Y.$$

If X and Y are test spaces, we make $X \oplus Y$ into a test space by letting $\mathbf{M}(X \oplus Y)$ equal the set $\{E \oplus F \mid E \in \mathbf{M}(X), F \in \mathbf{M}(Y)\}$. We can understand a test of the form $E \oplus F$ as a two-stage test: first, perform the classical two-outcome test $\{1, 2\}$ (by flipping a coin, say); if the result is 1, measure E , if the result is 2, measure F . A probability weight ω on $X \times Y$ corresponds to an arbitrary choice of a probability weight p on $\{1, 2\}$ and probability weights $\alpha \in \Omega(X)$ and $\beta \in \Omega(Y)$, by

$$\omega(1, x) = p(1)\alpha(x) \quad \text{and} \quad \omega(2, y) = p(2)\beta(y).$$

The weights p , α and β are uniquely determined by ω , so we can unambiguously write

$$\omega = t\alpha + (1 - t)\beta$$

In other words, $\Omega(X \oplus Y) = \Omega(X) \oplus \Omega(Y)$, whence, $\mathbf{E}(X \oplus Y) = \mathbf{E}(X) \oplus \mathbf{E}(Y)$.

Every discrete classical probabilistic model $(E, \Delta(E))$ is a direct convex sum of trivial models $(\{x\}, \delta_x)$ where $x \in E$ and $\delta_x(x) = 1$. In contrast, the basic quantum model $(X(\mathcal{H}), \Omega(\mathcal{H}))$ is irreducible. The more general models associated with matrix algebras arise as direct sums of irreducible quantum models.

2.4 Processes and Categories

In very broad terms, a *probabilistic theory* might be nothing more than a class of probabilistic models. But this usage is really much too broad. Part of the job of a theory is to tell us, not only which models represent “actual” systems, but also something about how such systems can change. In order to speak about systems changing, we need to introduce into the preceding formalism a notion of *process*. A natural place to start is with the idea of a mapping $\phi : \alpha \mapsto \phi(\alpha)$ taking states α of an initial (or input) system A to states of a final (output) system B . To allow for “lossy” processes or conditioning, we should permit $\phi(\alpha)$ to be a sub-normalized state of B when α is a normalized state of A . Finally, since randomizing the input state should randomize the output state in the same way, we should expect this ϕ to be an affine mapping. Thus, we model a process from A to B by an affine mapping $\phi : \Omega(A) \rightarrow \mathbf{E}(B)$ with $u_B(\phi(\alpha)) \leq 1$; or, what is the same thing, by a positive linear mapping $\phi : \mathbf{E}(A)^* \rightarrow \mathbf{E}(B)^*$ with $u_B \circ \phi \leq u_A$. We can interpret $u_B(\phi(\alpha))$ as the *probability* that ϕ occurs when the initial state is α — or, perhaps more accurately, as the probability that the process occurs, *if* initiated.

To every process $\phi : \mathbf{V}(B) \rightarrow \mathbf{V}(A)$, there corresponds a *dual process* $\tau = \phi^* : \mathbf{E}(A) \rightarrow \mathbf{E}(B)$, given by $\phi^*(a) = a \circ \phi$ for any $a \in \mathbf{E}(A)$. Operationally, to measure $\phi^*(a)$ on a state α , one first subjects the state α to the process ϕ , and then makes a measurement of the effect a . Note that $\tau(u)(\alpha) = u(\tau^*(\alpha))$ is the probability that the process $\tau^* = \phi$ occurs if the initial state is α . In what follows, it will often be more convenient mathematically to deal with these dual processes. In other words, to use physicists’ lingo, we’ll often work with the “Heisenberg” rather than the “Schrödinger” picture of processes.

Not every positive linear mapping $\mathbf{V}(A) \rightarrow \mathbf{V}(B)$ will generally count as a process. As remarked above, it is part of the job of a probabilistic theory to specify those that do. However, it seems reasonable to require that convex combinations of processes and composites of (composable) processes

also count as processes. It will also be convenient to assume that, for every pair of systems A and B , there is a *null process* that takes every state $\alpha \in \Omega(A)$ to the *zero state* $0 \in \mathbf{E}(B)$. It seems reasonable, also, that there exist a canonical *trivial* system I , corresponding to a test space with only a single outcome, 1, and a single test $\{1\}$. We then have $\mathbf{E}(I) = \mathbf{E}(I)^* = \mathbb{R}$. We can then require that, for every normalized state $\alpha \in \mathbf{V}(A)$, there exist a process $\mathbb{R} \rightarrow \mathbf{V}(A)$ of *preparation*, given by $1 \mapsto \alpha$, and, for every outcome $x \in X(A)$, a process $\mathbf{V}(A) \rightarrow \mathbb{R}$ of *registration*, sending $\alpha \in \mathbf{V}(A)$ to $\alpha(x)$. The dual process corresponding to the preparation of α is simply the state α itself, while the process dual to the registration of x is the linear mapping $\mathbb{R} \rightarrow \mathbf{E}(A)$ sending 1 to x . All of this suggests the following

Definition 8. A (state-complete) **probabilistic theory**⁹ is a category \mathcal{C} such that

- (1) Every object $A \in \mathcal{C}$ is a probabilistic model;
- (2) For all $A, B \in \mathcal{C}$, the set $\mathcal{C}(A, B)$ of morphisms $A \rightarrow B$ is a closed, convex subset of $L_+(\mathbf{E}(A), \mathbf{E}(B))$, containing the zero mapping, and with $\tau(u_A) \leq u_B$ for all $\tau \in \mathcal{C}(A, B)$;
- (3) There is a distinguished *trivial system* I with $\mathbf{E}(I) = \mathbb{R}$ and $X = \{1\}$, such that for every $A \in \mathcal{C}$, $X(A) \subseteq \mathcal{C}(I, A)$ and $\Omega(A) \subseteq \mathcal{C}(A, I)$.
- (4) The order unit $u_A \in \mathbf{E}(A)$ belongs to $\mathcal{C}(I, A)$.

From now on, we work in a fixed probabilistic theory \mathcal{C} of this kind. We write \mathcal{C}^* for the category having the same objects, but with morphisms $\mathcal{C}^*(A, B)$ the set of mappings $\phi = \tau^* : \mathbf{V}(B) \rightarrow \mathbf{V}(A)$ with $\tau \in \mathcal{C}(B, A)$. In effect, \mathcal{C} and \mathcal{C}^* offer, respectively, the “Heisenberg” and the “Schrödinger” picture of the same theory. Depending on context, we shall understand the word “process” to refer either to a morphism $\tau \in \mathcal{C}(A, B)$ for some $A, B \in \mathcal{C}$, or to the dual mapping $\phi = \tau^* : \mathbf{V}(B) \rightarrow \mathbf{V}(A)$.

Example 8. By a *standard finite-dimensional quantum theory*, we mean a category \mathcal{C} of probabilistic models (\mathbf{E}, X) where \mathbf{E} is the hermitian part of a finite-dimensional complex matrix algebra (a direct sum of algebras of the form $L(\mathcal{H})$), with trace-nonincreasing completely positive mappings as morphisms. In this formulation, classical probabilistic theories arise as the degenerate case in which all of the matrix algebras associated with systems in \mathcal{C} are commutative.

Reversible and Probabilistically Reversible Processes A process $\tau \in \mathcal{C}(A, B)$ is *reversible* iff it is invertible as a morphism in \mathcal{C} , i.e., there exists an inverse process $\tau^{-1} \in \mathcal{C}(B, A)$ with $\tau^{-1} \circ \tau = \text{id}_A$ and $\tau \circ \tau^{-1} = \text{id}_B$. In this case, τ is an order-automorphism $\mathbf{E}(A) \simeq \mathbf{E}(B)$, and $\tau^{-1} : \mathbf{E}(B) \simeq \mathbf{E}(A)$ is the inverse isomorphism. Moreover, for such a process, we have $\tau(u_A) = u_B$: by assumption, $\tau(u_A) \leq u_B$, and also $\tau^{-1}(u_B) \leq u_A$, whence, as τ preserves order, $u_B \leq \tau(u_A)$. Dually, a process $\phi \in \mathcal{C}^*(A, B)$ is reversible iff it has an inverse in $\mathcal{C}^*(B, A)$; equivalently, ϕ is invertible iff the dual process $\tau = \phi^*$ is invertible. In this case, we have $u_B \phi(\alpha) = 1$ for every normalized state $\alpha \in \Omega(A)$.

There is a weaker but very useful notion, which we shall call *probabilistic reversibility*. This is slightly easier to describe in terms of processes acting on states, rather than effects:

Definition 9. A process $\phi \in \mathcal{C}^*(A, B)$, is *probabilistically reversible* iff it is invertible as a linear mapping $\mathbf{V}(A) \rightarrow \mathbf{V}(B)$, with a positive inverse and if the inverse mapping ϕ^{-1} is a positive multiple of a process $\phi_o \in \mathcal{C}^*(B, A)$ — say, $\phi^{-1} = c\phi_o$ with $c > 0$.

Operationally, this means that there is some non-zero probability that $\phi_o \circ \phi$ will return the system to its original state. Indeed,

$$\phi_o(\phi(\alpha))(u_A) = c^{-1}\phi^{-1}(\phi(\alpha))(u_A) = c^{-1}\alpha(u_A) = c^{-1},$$

so this probability is exactly $1/c$. In particular, ϕ is reversible with probability one iff $c = 1$, so that ϕ^{-1} is a process in $\mathcal{C}^*(B, A)$ — in other words, ϕ is an reversible process.

⁹This definition differs from that of [17], most obviously in that objects are associated with effect spaces, rather than state spaces, but also in taking the test space $X(A)$ to be part of the structure of $A \in \mathcal{C}$.

We shall say that a process $\tau \in \mathcal{C}(A, B)$ is reversible with probability $1/c$ iff $\tau^* \in \mathcal{C}^*(A, B)$ is reversible. Obviously, the set of probabilistically reversible processes, in either $\mathcal{C}(A, A)$ or $\mathcal{C}^*(A, A)$, is a group, containing, but larger than, the group of all reversible processes on A .

Historical remarks: The representation of what we are calling probabilistic models in terms of an order-unit space and its dual goes back at least to the work of Davies and Lewis [27] and Edwards [29]. A good survey of the relevant functional analysis can be found in [2]. Test spaces — originally called “manuals” — were the basis for a generalized probability theory (and an associated “empirical logic”) developed in the 1970s and 80s by C. H. Randall and D. J. Foulis and their students. See [71] for a survey. Mathematically, of course, a test space is just a hypergraph; the current terminology serves only to reinforce the intended probabilistic interpretation.

3 Composition and Entanglement

Consider two systems, A and B , which are not interacting in any obvious, causal sense — for example, systems occupying space-like separated regions of space-time. In this situation, it seems reasonable to assume that what that can be *happen* to each system individually — the preparation of a state, the making of a measurement, etc. — can happen together, independently.

Another natural (albeit more contingent) requirement is a *no-signaling* condition, forbidding the transmission of information from A to B , or vice versa, by the mere decision to make one measurement rather than another on A , or on B . As we’ll see, the phenomenon of *entanglement*, one of the supposed hallmarks of quantum theory, is actually a rather generic feature of such “non-signaling” composite systems in non-classical probabilistic theories, whether “quantum” or otherwise. (Indeed, the phenomenon even arises in otherwise quite classical theories involving a restricted set of probability weights.)

3.1 Composites of Models

Suppose two parties — Alice and Bob, say — control, respectively, systems A and B , which occur as components of some composite system AB , but are still sufficiently isolated to be prepared and measured separately. At a very minimum, we would expect Alice’s making a measurement, E , on here part of the composite system, and Bob’s making a measurement, F , on his part, *constitutes* the making of a measurement on the combined system. We would also expect that states of the two component systems can be prepared independently. Formalizing these requirements, we arrive at the following:

Definition 10. A composite of two probabilistic models A and B is a model AB , together with a mapping

$$X(A) \times Y(B) \rightarrow X(AB) : (x, y) \mapsto xy$$

such that

- (i) for all tests $E \in \mathcal{M}$ and $F \in B$, the product test $EF := \{xy | x \in E, y \in F\}$ belongs to $\mathcal{M}(AB)$; and
- (ii) for all states $\alpha \in \Omega(A)$ and $\beta \in \Omega(B)$, there exists a **unique** [?] state $\alpha \otimes \beta \in \Omega(AB)$ with $(\alpha \otimes \beta)(xy) = \alpha(x)\beta(y)$.

Remarks: There are several ways in which we might plausibly weaken this definition. For instance, we might require only that the product outcome xy be an *effect* in $\mathbf{E}(AB)_+$, and the set EF , an observable, but not necessarily a test, of AB .¹⁰ Such possibilities are worth bearing in mind. However, for the purposes of this survey, it seems reasonable to use the more restrictive, but therefore

¹⁰More radically, one might consider models of systems interacting in such a way that the making of a particular measurement, or the preparation of a particular state, on one component, *precludes* the making of certain measurements, or the preparation of certain states, on the other component. Mathematically, such situations are certainly possible.

simpler, definition above. Note in (ii) we require only the existence, but not the uniqueness [??], of product states (where a product state for α and β is defined as a state γ with $\gamma(xy) = \alpha(x)\beta(y)$, and a product state *tout court* as one that is a product state for a pair of states α and β).

The injectivity of the mapping $x, y \mapsto xy$ in condition (i) allows us to identify $X(A) \times X(B)$ with the Let us write

$$X(A)X(B) := \{xy | x \in X, y \in Y\}$$

for the square of *product outcomes* in Z . With a slight abuse of notation, we may write $\mathcal{M}(A) \times \mathcal{M}(B)$ for the test space consisting of product tests EF . Condition (i) asserts that $\mathcal{M}(A) \times \mathcal{M}(B)$ is contained in $\mathcal{M}(AB)$, so every state in $\Omega(AB)$ restricts to a state ω_o on the former. Where the restricted state ω_o *determines* the global state ω — that is, where the set $X(A)X(B)$ of product outcomes is state-separating — we say that the composite is *locally tomographic*. In this setting, the joint probabilities of outcomes of measurements on the component systems A and B , completely determine the state of the composite.¹¹ This is a reasonable, but also a rather strong, restriction. Indeed, while composites in standard complex QM are locally tomographic, this is not the case for real or quaternionic QM. We'll return to this matter below.

Example 9 (Composite quantum models). If $A(\mathcal{H})$ and $A(\mathbf{K})$ are two quantum-mechanical models, associated with finite-dimensional Hilbert spaces \mathcal{H} and \mathbf{K} , respectively, let

$$A(\mathcal{H})A(\mathbf{K}) = A(\mathcal{H} \otimes \mathbf{K})$$

the model associated with $\mathcal{H} \otimes \mathbf{K}$. That is, $\mathcal{M}(\mathcal{H} \otimes \mathbf{K})$ consists of orthonormal bases for $\mathcal{H} \otimes \mathbf{K}$, while $\Omega(\mathcal{H} \otimes \mathbf{K})$ consists of density operators on $\mathcal{H} \otimes \mathbf{K}$. If $x \in \mathcal{H}$ and $y \in \mathbf{K}$ are unit vectors, then $x \otimes y$ is a unit vector in $\mathcal{H} \otimes \mathbf{K}$. It is easy to check that $x, y \mapsto x \otimes y$ makes $A(\mathcal{H} \otimes \mathbf{K})$ into a composite in the sense of the preceding definition.

3.2 Non-Signaling Composites and Entanglement

The very broad definition of a composite system given above leaves room for situations in which the probability of Bob's obtaining an outcome y will depend on which test $E \in \mathcal{M}(A)$ Alice chooses to measure. This is plausible only in scenarios in which Alice's measurements are able physically to disturb Bob's system. If we wish to model composites in which the two systems A and B are sufficiently isolated from one another that this kind of remote disturbance is ruled out — the obvious situation being one in which A and B are spacelike separated — then we must impose a further constraint.

Definition 11. A probability weight ω on $\mathcal{M}(A) \times \mathcal{M}(B)$ is non-signaling iff it has well-defined marginal (or reduced) states, in the sense that

$$\omega_1(x) := \sum_{y \in F} \omega(xy) \quad \text{and} \quad \omega_2(y) := \sum_{x \in E} \omega(xy)$$

are independent of the choice of tests $E \in \mathcal{M}(A)$, $F \in \mathcal{M}(B)$.

If $\omega \in \Omega(AB)$ is non-signaling, then for every $y \in X(B)$ and $x \in X(A)$, we can define the *conditional states* $\omega_{1|y}$ and $\omega_{2|x}$ on A and B , respectively, by

$$\omega_{1|y}(x) := \frac{\omega(xy)}{\omega_2(y)} \quad \text{and} \quad \omega_{2|x}(y) := \frac{\omega(xy)}{\omega_1(x)}.$$

These are well-defined probability weights on $\mathcal{M}(A)$ and $\mathcal{M}(B)$, respectively. It would seem reasonable to include them in the state spaces of A and B . Therefore, we adopt the following language:

¹¹Barrett [19] calls this the *global state hypothesis*; the term *locally tomographic* seems to have become more standard.

Definition 12. A *non-signaling composite* of A and B is a composite AB in which all states are non-signaling, and all conditional states belong to the designated state spaces of A and B — that is, $\omega_{2|x} \in \Omega(B)$ and $\omega_{1|y} \in \Omega(A)$ for all $x \in X(A)$ and $y \in X(B)$.

This has a strong consequence [67]:

Lemma 3 (Bi-Linearization). *Let AB be a non-signaling composite of A and B . Then every state $\omega \in \Omega(AB)$ extends uniquely to a bilinear form on $\mathbf{E}(A) \times \mathbf{E}(B)$.*

Proof: For every $x \in X(A)$, define $\hat{\omega}(x) \in \mathbb{R}^{X(B)}$ by $\hat{\omega}(x)(y) = \omega(x, y)$. Notice that $\omega_{2|x} = \hat{\omega}(x)/\omega_1(x)$. Since the conditional state $\omega_{2|x}$ belongs to $\Omega(B)$, we have $\hat{\omega}(x) \in \mathbf{V}(B) = \mathbf{E}(B)^*$, with $\sum_{x \in E} \hat{\omega}(x) = \omega_2$. Dualizing (and remembering that $\mathbf{E}(A)$ is finite-dimensional), we have a linear mapping $\hat{\omega}^* : \mathbf{E}(B) \rightarrow \mathbb{R}^{X(A)}$. Now, $\hat{\omega}^*(y) = \omega_{1|y}/\omega_2(y)$; the latter belongs to $\Omega(A)$, so $\hat{\omega}^*(y) \in \mathbf{V}(A) = \mathbf{E}(A)^*$ for every $y \in X(B)$. Since $X(B)$ spans $\mathbf{E}(B)$, it follows that the range of $\hat{\omega}^*$ lies in $\mathbf{V}(A)$, i.e., we can regard $\hat{\omega}^*$ as a linear mapping $\mathbf{E}(A) \rightarrow \mathbf{V}(B) = \mathbf{E}(B)^*$. Equivalently, we have a bilinear form $\mathcal{B}_\omega(a, b) = \hat{\omega}^*(b)(a)$, which evidently satisfies $\mathcal{B}_\omega(x, y) = \omega(xy)$ for all $x \in X(A), y \in X(B)$. Since $X(A)$ and $X(B)$ span $\mathbf{E}(A)$ and $\mathbf{E}(B)$, the form \mathcal{B}_ω is uniquely determined by this property. \square

It follows that, for a non-signaling composite, the mapping $X(A) \times X(B) \rightarrow X(AB) : x, y \mapsto xy$ gives rise to a linear mapping $\otimes : \mathbf{E}(A) \otimes \mathbf{E}(B) \rightarrow \mathbf{E}(AB)$, with $\omega(x \otimes y) = \mathcal{B}_\omega(x, y) = \omega(xy)$ for every $\omega \in \mathbf{E}(AB)^*$. The composite AB is locally tomographic iff this mapping is surjective.

Corollary 1. *A non-signaling composite AB of models A and B is locally tomographic iff $\mathbf{E}(AB) \simeq \mathbf{E}(A) \otimes \mathbf{E}(B)$, that is, $\dim(\mathbf{E}(AB)) = \dim(\mathbf{E}(A)) \dim(\mathbf{E}(B))$.*

Lemma 3 allow us to extend the definition of conditional states to arbitrary effects, setting

$$\omega_{1|b}(a) = \omega(a \otimes b)/\omega(u \otimes b) \quad \text{and} \quad \omega_{2|a}(b) = \omega(a \otimes b)/\omega(a \otimes u)$$

for arbitrary effects $a \in \mathbf{E}(A)$ and $b \in \mathbf{E}(B)$ (with the usual proviso about division by zero). The following bipartite version of the law of total probability is easily verified:

Lemma 4 (Law of Total Probability). *Let AB be a non-signaling composite of A and B ; let ω be any state on AB , and let E and F be any two observables on $\mathbf{E}(A)$ and $\mathbf{E}(B)$, respectively, then*

$$\omega_2 = \sum_{a \in E} \omega_1(a) \omega_{2|a} \quad \text{and} \quad \omega_1 = \sum_{b \in F} \omega_2(b) \omega_{1|b}$$

Corollary 2. *Let AB be a non-signaling composite of A and B , and let ω be a pure state of AB . If the marginal state ω_2 is pure, then ω_1 is also pure, and $\omega = \omega_1 \otimes \omega_2$.*

Proof: It is easy to see that, if a product state $\omega = \omega_1 \otimes \omega_2$ is pure, then both marginals must be pure. Now suppose that one marginal state — say, ω_2 — is pure. Since $\omega_2 = \sum_{x \in E} \omega_1(x) \omega_{2|x}$, and the conditional states $\omega_{2|x}$ belong to $\mathbf{V}(B)$, it follows that for every $x \in E$ with $\omega_1(x) > 0$, we must have $\omega_{2|x} = \omega_2$, so that $\omega(xy) = \omega_1(x) \omega_2(y)$ for every such x . The same result holds trivially if $\omega_1(x) = 0$, so we have $\omega(xy) = \omega_1(x) \omega_2(y)$ for all choices of x and y . It follows that $\omega = \omega_1 \otimes \omega_2$. \square

Definition 13. A state ω on AB is *separable* iff it is a mixture of product states, that is, $\omega = \sum_i t_i \alpha_i \otimes \beta_i$ where $t_i \geq 0$ and $\sum_i t_i = 1$. A state *not* of this form is said to be *entangled*.

Using this language, the preceding Corollary gives us

Corollary 3. *If AB is a non-signaling composite of models A and B , and ω is an entangled state of AB , then both ω_1 and ω_2 are mixed.*

This is often regarded as the hallmark of entangled *quantum* states; but, as we see, it is really a quite general possibility arising in any non-classical probabilistic setting. Of course, one can still ask at this point whether entangled states *exist* in any generality, once one leaves the confines of quantum theory. However, as we'll see in Section 3.4 below, there is a sense in which *most* non-signaling composites of non-classical models admit entangled states.

The CHSH Inequality Let AA be a non-signaling composite of two copies of A . For any $a, b \in \mathbf{E}(A)$ with $-u_A \leq a, b \leq u_A$, let $a' = u_A - a$ and $b' = u_A - b$. For any state ω in AA , define

$$S(\omega; a, b) = \omega(a, b) + \omega(a, b') + \omega(a', b) - \omega(a', b').$$

This is called the CHSH (Clauser-Horn-Shimony-Holt) parameter associated with ω , a and b . of a bipartite If ω is a product state, then $S \leq 2$ for all choices of a and b ; as S is affine in ω , it follows that $S \leq 2$ for all separable states. For entangled states it can be larger. A priori, the upper bound for S is 4, and this is achieved, for example, if A is the “square bit” of example 3. However, for bipartite quantum states, the upper bound is much lower. As pointed out by Tsirel'son [64], $S \leq 2\sqrt{2}$ for any quantum bipartite state and any effects a and b . A great deal of work has gone into trying to find a deeper explanation for this bound. [3, 53]. In section 4, we will return to this matter.

Conditioning Maps and Isomorphism States If ω is any non-signaling state on AB , then the associated bilinear form \mathcal{B}_ω on $\mathbf{E}(A) \times \mathbf{E}(B)$ gives us a positive linear mapping

$$\widehat{\omega} : \mathbf{E}(A) \rightarrow \mathbf{E}(B)^*$$

defined by

$$\widehat{\omega}(a)(b) = \omega(a \otimes b)$$

for all $a \in \mathbf{E}(A)$ and $b \in \mathbf{E}(B)$. Notice that $\widehat{\omega}(a) = \omega_1(a)\omega_2|_a$. Accordingly, we think of $\widehat{\omega}(a)$ as an *un-normalized conditional state* of B given the effect $a \in \mathbf{E}(A)$, and refer to $\widehat{\omega}$ as the *conditioning map* associated with ω . Of course, there is also a conditioning map running in the opposite direction. In fact, this is just the adjoint of $\widehat{\omega}$; that is, $\widehat{\omega}^*(b)(a) = \widehat{\omega}(a)(b) = \omega(a, b)$ for all effects $a \in \mathbf{E}(A)$ and $b \in \mathbf{E}(B)$.

There is a dual construction for effects. An effect $f \in \mathbf{E}(AB)$ defines a positive bilinear form on $\mathbf{V}(A) \times \mathbf{V}(B)$ by $(\alpha, \beta) \mapsto f(\alpha \otimes \beta)$. This, in turn, yields a positive linear mapping

$$\widehat{f} : \mathbf{V}(A) \rightarrow \mathbf{V}(B)^* = \mathbf{E}(B)$$

given by $\widehat{f}(\alpha)(\beta) = f(\alpha \otimes \beta)$. We call \widehat{f} the *co-conditioning map* associated with f .

Definition 14. Let AB be a non-signaling composite of A and B . An *isomorphism state* on AB is a state $\omega \in \Omega(AB)$ such that the conditioning map $\widehat{\omega} : \mathbf{E}(A) \rightarrow \mathbf{V}(B)$ is an order-isomorphism. Dually, an *isomorphism effect* is an effect $f \in \mathbf{E}(AB)$ such that the co-conditioning map $\widehat{f} : \mathbf{V}(A) \rightarrow \mathbf{E}(B)$ is an order-isomorphism.

Evidently, the inverse of an isomorphism state is a multiple of an isomorphism effect, and vice versa. This point will be important in the discussion of teleportation protocols below. If there exists an isomorphism state on a composite AA of A with itself, then we have $\mathbf{E}(A) \simeq \mathbf{V}(A) = \mathbf{E}(A)^*$.¹² More generally, we shall say that A is *weakly self-dual* iff there exists an order-isomorphism $\mathbf{E}(A) \simeq \mathbf{V}(A)$ (equivalently: an isomorphism state in $A \otimes_{\max} A$). Although this is a strong constraint on the structure of a probabilistic model, it is nevertheless satisfied by many examples that are neither quantum nor classical. For example, the models associated with state spaces that are regular 2-dimensional polytopes — that is, regular n -gons — are weakly self-dual.

As we'll discuss further in Section 5, quantum models satisfy a much stronger form of self-duality: not only does there exist an order-isomorphism $\mathbf{V}(\mathcal{H}) \simeq \mathbf{E}(\mathcal{H})$, but this is given by an inner product on $\mathbf{E}(\mathcal{H}) = \mathbf{L}(\mathcal{H})$, namely, $a \mapsto \text{Tr}(a \cdot)$.

¹²The converse is not quite true: an order-isomorphism $\mathbf{E}(A) \simeq \mathbf{V}(A)$ defines a non-signaling state on $A \otimes_{\max} B$ [def.], but need not correspond to a state of AB .

Proposition 4 ([14]). *Let A and B be irreducible, and let AB be any locally-tomographic, non-signaling composite of A with B . Then any isomorphism state in AB is pure in $\Omega(AB)$, and any isomorphism effect is extremal in $\mathbf{E}(AB)_+$.*

If A and B are not irreducible, an isomorphism state on AB need not be pure. For example, if $A = B = (E, \Delta(E))$, then any state uniformly correlating A and B — say $\omega(x, x) = 1/|E|$ and $\omega(x, y) = 0$ for $x \neq y$ — is an isomorphism state, but will be pure only if $|E| = 1$.

3.3 Quantum Composites

This is a good place at which to pause for a second and more detailed look at quantum-mechanical composites. As noted earlier in Example 2, the mapping $X(\mathcal{H}) \times X(\mathbf{K}) \mapsto X(\mathcal{H} \otimes \mathbf{K})$ given by $x, y \mapsto x \otimes y$ turns $A(\mathcal{H} \otimes \mathbf{K})$ into a composite of the models $A(\mathcal{H})$ and $A(\mathbf{K})$. This mapping extends to the bilinear mapping

$$\mathbf{E}(\mathcal{H}) \times \mathbf{E}(\mathbf{K}) = \mathbf{L}_h(\mathcal{H}) \times \mathbf{L}_h(\mathbf{K}) \rightarrow \mathbf{L}_h(\mathcal{H} \otimes \mathbf{K}) = \mathbf{E}(\mathcal{H} \otimes \mathbf{K}),$$

that sends $a, b \in \mathbf{L}_h(\mathcal{H}) \times \mathbf{L}_h(\mathbf{K})$ to the operator $a \otimes b$ on $\mathcal{H} \otimes \mathbf{K}$ (given by $(a \otimes b)(x \otimes y) = ax \otimes by$ for all $x \in \mathcal{H}, y \in \mathbf{K}$). Hence, by Lemma 3, $A(\mathcal{H} \otimes \mathbf{K})$ is a *non-signaling* product of $A(\mathcal{H})$ and $A(\mathbf{K})$.

Conditioning Let \mathcal{H} be a complex Hilbert space. For any vectors $x, y \in \mathcal{H}$, let $x \odot y$ denote the rank-one operator on \mathcal{H} given by $(x \odot y)z = \langle z, y \rangle x$. (In Dirac notation, this is $|x\rangle\langle y|$.) If x is a unit vector, then $x \odot x = P_x$, the orthogonal projection operator associated with x .

The mapping $x, y \mapsto x \odot y$ is sesquilinear, that is, linear in its first, and conjugate linear in its second, argument; it therefore extends to a linear mapping $\mathcal{H} \otimes \overline{\mathcal{H}} \rightarrow \mathbf{L}(\mathcal{H})$, where $\overline{\mathcal{H}}$ is the conjugate space of \mathcal{H} , taking any vector $v = \sum_i t_i x_i \otimes \overline{y}_i$ to the corresponding operator $\widehat{v} := \sum_i t_i x_i \odot y_i$. It is easy to see that this is injective and hence, on dimensional grounds, an isomorphism. It is useful to note that

$$\langle \widehat{v}(x), y \rangle = \langle v, y \otimes \overline{x} \rangle$$

for all $x, y \in \mathcal{H}$. Hence, if v is any unit vector in $\mathcal{H} \otimes \overline{\mathcal{H}}$, the corresponding pure state $\omega = \alpha_v$ of $A(\mathcal{H} \otimes \overline{\mathcal{H}})$ assigns joint probabilities to outcomes $x \in X(\mathcal{H})$ and $\overline{y} \in X(\overline{\mathcal{H}})$ by

$$\omega(x, \overline{y}) = |\langle v, x \otimes \overline{y} \rangle|^2 = |\langle \widehat{v}(y), x \rangle|^2$$

so that the conditional state $\omega_{2|\overline{y}}$ is exactly the pure state associated with the unit vector $\widehat{v}(y)/\|\widehat{v}(y)\|$. (The fact that conditioning a pure bipartite quantum state by a measurement outcome always leads to a pure state — the *pure conditioning property* — is rather special, and has been exploited in [23, 70].)

Purification and Correlation Suppose now that α is a state on $A(\mathcal{H})$, represented by a density operator W on \mathcal{H} with spectral resolution

$$W = \sum_{x \in E} \lambda_x P_x = \sum_{x \in E} \lambda_x x \odot x$$

where E is an orthonormal basis for \mathcal{H} and $\sum_{x \in E} \lambda_x = \text{Tr}(W) = 1$. Functional calculus gives us $W^{1/2} = \sum_{x \in E} \lambda_x^{1/2} x \odot x$. We can interpret this as a unit vector in $\mathcal{H} \otimes \overline{\mathcal{H}}$, namely

$$\Psi_W := \sum_{x \in E} \lambda_x^{1/2} x \otimes \overline{x}. \quad (1)$$

This, in turn, defines a bipartite state on the composite quantum system $A\overline{A} := A(\mathcal{H} \otimes \overline{\mathcal{H}})$. The marginal, or reduced, state of the first component system is given by

$$\omega_1(a) = \text{Tr}(P_{\Psi_W}(a \otimes \mathbf{1}_{\overline{\mathcal{H}}})) = \langle (a \otimes \mathbf{1}_{\overline{\mathcal{H}}})\Psi_W, \Psi_W \rangle = \text{Tr}(Wa)$$

so the pure state corresponding to Ψ_W is a *dilation* of the given mixed state W . Now observe that if $u, v \in X(\mathcal{H})$ with $u \perp v$, then we have

$$\langle \Psi_W, u \otimes \bar{v} \rangle = \sum_{x \in E} \lambda_x^{1/2} \langle x, u \rangle \langle \bar{x}, v \rangle = 0.$$

Evidently, the pure state ω corresponding to Ψ_W sets up a *perfect correlation* between $E \in \mathbf{M}(\mathcal{H})$ and the corresponding test $\bar{E} = \{\bar{x} | x \in E\} \in \mathbf{M}(\bar{\mathcal{H}})$, with

$$\omega(x, \bar{x}) = |\langle \Psi_W, x \otimes \bar{x} \rangle|^2 = |\lambda_x^{1/2}|^2 = \lambda_x.$$

An especially interesting case arises when α is the maximally mixed state, i.e., when $W = \mathbf{1}/n$ (where $n = \dim(\mathcal{H})$). Then Ψ_W is *independent* of the choice of E (since every orthonormal basis of \mathcal{H} is an eigenbasis for $\mathbf{1}$). Hence, Ψ_W simultaneously correlates *every* test $E \in \mathbf{M}(\mathcal{H})$ with its counterpart in $\mathbf{M}(\bar{\mathcal{H}})$. Moreover, the correlation is *uniform*, in that the probabilities of correlated pairs $x \otimes \bar{x}$ of outcomes is uniformly $1/n$. As we'll see later, the existence of such a uniformly correlating state between two isomorphic systems has interesting consequences.

Local Tomography If \mathcal{H} and K are real or complex Hilbert spaces of dimensions m and n , respectively, As was remarked above, $A(\mathcal{H} \otimes K)$ is a non-signaling composite of $A(\mathcal{H})$ and $A(K)$. It is easily checked that $\dim E(A) = \dim \mathcal{L}_h(\mathcal{H}) = m^2$ if \mathcal{H} is complex and $(m^2 + m)/2$ if \mathcal{H} is real. Hence, the dimension of the real vector space $E(\mathcal{H} \otimes K) = \mathcal{L}_h(\mathcal{H} \otimes K)$ of Hermitian operators is $(mn)^2 = m^2n^2$, so in fact $\mathcal{L}_h(\mathcal{H} \otimes K) = \mathcal{L}_h(\mathcal{H}) \otimes \mathcal{L}_h(K)$, and the composite system is locally tomographic. On the other hand, if \mathcal{H} and K are real, the dimension of $\mathcal{L}_h(\mathcal{H} \otimes K)$ is $((mn)^2 - mn)/2 + mn = ((mn)^2 + mn)/2$, while the product of the dimensions of $\mathcal{L}_h(\mathcal{H})$ and $\mathcal{L}_h(K)$ is

$$\frac{(m^2 + m)}{2} \cdot \frac{(n^2 + n)}{2} = \frac{m^2n^2 + m^2n + mn^2 + mn}{4}.$$

This is strictly less than $(m^2n^2 + mn)/2$, which in turn is less than $(mn)^2$, so in this case, $E(AB)$ is strictly larger than $E(A) \otimes E(B)$. Thus, for real Hilbert spaces \mathcal{H} and K , the standard composite $\mathbf{M}(\mathcal{H} \otimes K)$ is *not* locally tomographic. (Neither do we have local tomography for quaternionic Hilbert spaces, though here, one needs to be more careful about the formulation of the relevant tensor products. See [6] and [44] for more details.)

3.4 Maximal and Minimal Tensor Products

Let AB be a non-signaling composite of two systems A and B . As noted above, if AB is locally tomographic, then $E(AB) \simeq E(A) \otimes E(B)$ as vector spaces. In this section, we consider more closely the possibilities for such a composite.

As we saw earlier, any non-signaling state ω on any composite system AB is associated with a bilinear form on $E(A) \times E(B)$. If AB is locally tomographic, then we can identify ω with this form. We then see that there are two extreme possibilities for the set of states on a locally tomographic composite AB : maximally, we may include *all* positive, normalized bilinear forms on $E(A) \times E(B)$; minimally, we may restrict our attention to the closed convex hull of the product states.

Definition 15. Let E and F be any two finite-dimensional ordered vector spaces. The *minimal tensor cone* on $E \otimes F$ is the cone generated by pure tensors $a \otimes b$ with $a \in E_+$ and $b \in F_+$. The *maximal tensor cone* is the cone of all tensors $\tau \in E \otimes F$ such that $\tau(\omega) \geq 0$ for all $\omega \in \mathcal{G}_+(E, F)$. These two cones give us two different ordered tensor products, which we denote by $E \otimes_{\min} F$ and $E \otimes_{\max} F$, respectively.

It is not difficult to see that (in finite dimensions) we have

$$(E \otimes_{\min} F)^* = E^* \otimes_{\max} F^* \text{ and } (E \otimes_{\max} F)^* = E^* \otimes_{\min} F^*.$$

Let AB be any locally tomographic composite of models A and B . Then the set $X(A)X(B)$ of product outcomes in $E(AB) \simeq E(A) \otimes E(B)$ generates exactly the minimal tensor cone in

$\mathbf{E}(A) \otimes \mathbf{E}(B)$. It follows that the cone of un-normalized non-signaling states on $\mathbf{M}(A) \times \mathbf{M}(B)$ [defined?] is exactly the maximal tensor cone in $\mathbf{V}(A) \otimes \mathbf{V}(B)$. Dually, the set of product states generates the minimal tensor cone in $\mathbf{V}(A) \otimes \mathbf{V}(B)$.

Definition 16. Thus, we may define the *minimal tensor product* of A and B to be the model $A \otimes_{\min} B = (\mathbf{E}(A) \otimes_{\min} \mathbf{E}(B), X(A) \times X(B))$. By the *maximal tensor product* of A and B we mean the model $(\mathbf{E}(A) \otimes_{\max} \mathbf{E}(B), X(A) \otimes_{\max} X(B))$, where the test space $X(A) \otimes_{\max} X(B)$ is the maximal test space for $\mathbf{E}(A) \otimes_{\max} \mathbf{E}(B)$.

These choices of these two test spaces are dictated by the desire to have the following

Proposition 5. *If AB is any locally tomographic composite of A and B , then we have embeddings $A \otimes_{\min} B \rightarrow AB \rightarrow A \otimes_{\max} B$. We also have, dually, $\Omega(A \otimes_{\max} B) \leq \Omega(AB) \leq \Omega(A \otimes_{\min} B)$.*

Thus, $A \otimes_{\min} B$ is the smallest possible locally tomographic composite of A and B , in the sense of having the fewest possible effects. Dually, $\mathbf{E}(A \otimes_{\min} B)^* = \mathbf{E}(A)^* \otimes_{\max} \mathbf{E}(B)^*$ has the *largest* possible state space among locally tomographic composites. One might say, roughly speaking, that $A \otimes_{\min} B$ admits *no* entanglement between effects, and, consequently, admits all possible entangled states. At the other extreme, $A \otimes_{\max} B$ admits every possible entangled bipartite effect and, in consequence, admits no entanglement of states.

If $\Omega(A)$ or $\Omega(B)$ is a simplex, then it is easy to show that $\mathbf{V}(A) \otimes_{\max} \mathbf{V}(B) \simeq \mathbf{V}(A) \otimes_{\min} \mathbf{V}(B)$ and $\mathbf{E}(A) \otimes_{\max} \mathbf{E}(B) \simeq \mathbf{E}(A) \otimes_{\min} \mathbf{E}(B)$. Thus, a classical system admits no entangled states or effects in any non-signaling composite with another system. There is a partial converse:

Theorem 6 ([52]). *The following are equivalent:*

- (a) $\Omega(A \otimes_{\max} B)$ contains no entangled state for any model B ,
- (b) $\Omega(A \otimes_{\max} B)$ contains no entangled state, where B is the square bit (Example....),
- (c) $\Omega(A)$ is a simplex.

It follows that any *non*-classical system A — one with a non-simplicial state space — will admit *some* locally tomographic, non-signaling composite AB that admits entangled states. In this sense, entanglement is a highly generic phenomenon in non-classical probability theory.

3.5 Monoidal Probabilistic Theories

Earlier, we decided to represent a probabilistic *theory* as a category of probabilistic models with positive mappings as morphisms. It is not unreasonable to require that, if A, B and C are three systems, we should be able to form tripartite composites $(AB)C$ and $A(BC)$. We'd perhaps like to require that these be the same, i.e., that we have an *associative* rule of composition. This is not a trivial requirement — one can readily imagine situations in which the composition of systems might not be associative¹³ — but it is a natural one.

A *symmetric monoidal category* is a category \mathcal{C} , equipped with a bi-functor $\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$, such that for all $A, B, C, D \in \mathcal{C}$,

$$A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C \quad \text{and} \quad A \otimes B \simeq B \otimes A$$

by means of natural isomorphism $\alpha_{A,BC}$ and $\sigma_{A,B}$ belonging to \mathcal{C} ; and also equipped with a *tensor unit*, I , and natural isomorphisms

$$I \otimes A \simeq A \simeq A \otimes I$$

¹³Consider, for instance, the case of

(Farmer \otimes Hen) \otimes Fox *vs.* Farmer \otimes (Hen \otimes Fox).

This point of view has been extensively developed in the the categorical semantics for quantum theory developed by Abramsky-Coecke and Selinger [1, 24, 59], and also in the work of Baez and his students [6, 7].

Definition 17. *A monoidal probabilistic theory is a probabilistic theory \mathcal{C} , equipped with a rule of composition $A, B \mapsto AB$ assigning, to each pair of models $A, B \in \mathcal{C}$, a composite AB in the sense of Definition 10, and making \mathcal{C} a symmetric monoidal category. We shall say that \mathcal{C} is non-signaling, respectively locally tomographic, iff AB is non-signaling or locally tomographic for every pair $A, B \in \mathcal{C}$.*

This definition implies that, for all $A, B \in \mathcal{C}$ and all states $\alpha \in \Omega(A)$, $\beta \in \Omega(B)$, there is a distinguished product state $\alpha \otimes \beta$ with $(\alpha \otimes \beta)(xy) = \alpha(x)\beta(y)$ for all $x \in X(A)$, $y \in X(B)$. Similarly, for any (dual) processes $\tau_1 \in \mathcal{C}(A)$ and $\tau_2 \in \mathcal{C}(B)$, there exists a process $(\tau_1 \otimes \tau_2) \in \mathcal{C}(AB)$ with $(\tau_1 \otimes \tau_2)(a \otimes b) = \tau_1(a) \otimes \tau_2(b)$ for all effects $a \in \mathbf{E}(A)$ and $b \in \mathbf{E}(B)$.

Finite-dimensional classical and quantum probability theory are both monoidal with respect to their usual rules of composition. The minimal and maximal tensor products are each naturally associative, and hence make the category of *all* probabilistic models into a monoidal probabilistic theory; but neither is entirely satisfactory: the former provides for entangled states, but does not permit entangled effects, while the latter provides for entanglement between effects, but allows none between states. That a probabilistic theory support a *single* “tensor product” that accommodates entanglement of both states and effects, is a non-trivial constraint. To be sure, one might consider probabilistic theories equipped with more than one rule of composition; however, the interactions among different non-signaling compositions on a given theory can be very delicate. It therefore seems reasonable to begin by investigating the simpler possibilities for a theory equipped with a single privileged, monoidal rule of composition. Accordingly, *in the balance of this paper, we work in a monoidal probabilistic theory \mathcal{C} .*

Historical Remarks Tensor products of compact convex sets or of order-unit spaces were studied in a number of papers in the late 1960s, notably that of Namioka and Phelps [52]. The fact that the marginal of an entangled pure state must be a mixed state already appears there, albeit not in these terms, as do the definitions of what we are calling the maximal and minimal tensor products. Our treatment composite systems derives from that of by Foulis and Randall [34, 44]. Some first attempts to understand probabilistic theories as symmetric monoidal categories of probabilistic models can be found in [17, 15]; work in this direction is ongoing.

4 Post-Classical Information Processing

As we’ve seen, entangled bipartite states and effects arise very naturally, not only in quantum theory, but in almost any context in which we form non-signaling composites of non-classical systems. While this observation goes back at least to [44, 43] in the late 1980s, it remained unexploited. Entanglement lies at the heart of quantum information theory, so it natural to wonder to what extent quantum information-theoretic results carry over to other non-classical settings. It turns out that a great many such results do have analogues for probabilistic theories that are far more general than quantum mechanics. While the exploration of this post-classical information theory is still in its infancy, it has already shed considerable light on the scope and meaning of several key quantum-informational results.

In this section, we review in some detail two of these. The first is the no-cloning theorem, and its generalization, the no-broadcasting theorem. These hold in *any* finite-dimensional theory having a state space that is not a simplex. The second is the existence of a teleportation protocol, or, a bit more generally, of an entanglement-swapping protocol. Here, some restrictions need to be made, but they are of moderate strength. For example, any monoidal probabilistic theory in which individual systems are *weakly self-dual*, and composites include isomorphism states ω and effects f corresponding to isomorphisms $\hat{\omega}$, \hat{f} witnessing the weak self-duality, supports a certain kind of

teleportation. Moreover, when viewed in this generality, teleportation loses most of its mystery: it is simply a form of classical conditioning, one which appears startling only owing to the appearance of isomorphism states.

4.1 Cloning and broadcasting

To *clone* a state of a system A means, very broadly, to produce two *independent* copies of that state by means of some physical process. In the present formalism, if the initial state belongs to a system A , this would require a positive linear mapping

$$\phi : \mathbf{V}(A) \rightarrow \mathbf{V}(AA)$$

such that $\phi(\alpha) = \alpha \otimes \alpha$. There is no difficulty producing such a mapping: indeed, the constant mapping $\Omega(A) \rightarrow \Omega(AA)$ given by $\beta \mapsto \alpha$ for all $\beta \in \Omega(A)$ is affine, and hence, extends uniquely to a positive linear mapping $\mathbf{V}(A) \rightarrow \mathbf{V}(AA)$. However, this mapping is (highly!) state-dependent. One might ask whether one could *jointly* clone a collection of states, say, $\alpha_1, \dots, \alpha_n$. That is: given such a set of states, can one find a *single*, norm-nonincreasing, positive linear mapping $\mathbf{V}(A) \rightarrow \mathbf{V}(AA)$ that clones them all, in the sense that $\phi(\alpha_i) = \alpha_i \otimes \alpha_i$ for all i ?

If the states α_i are jointly distinguishable, the answer is yes. If $\{a_i\}$ is an observable on A with $\alpha_i(a_i) = 1$; then the mapping

$$\phi(\beta) = \sum_i \beta(a_i) \alpha_i \otimes \alpha_i$$

does the trick. The *no-cloning theorem* is essentially the converse: if there exists a single process that will clone all of the states $\alpha_1, \dots, \alpha_n$, then there exists an observable that distinguishes them. In the case of a discrete classical model, where all pure states are jointly distinguishable, this is no restriction on the clonability of pure states; but quantum pure states, which are not jointly distinguishable, are in general not jointly clonable.

The quantum no-cloning theorem was first proved, independently, by Wootters and Zurek [73] and by Dieks [25]. That the same result holds for arbitrary probabilistic theories is proved in [9]. We omit the proof here, but the idea is simple: if we can clone each of the states $\alpha_1, \dots, \alpha_n$ with a single mapping, then by iterating this process, we can create arbitrarily large ensembles of independent copies of an unknown state $\alpha \in \{\alpha_1, \dots, \alpha_n\}$ and, by making measurements on this ensemble, we can use statistics to distinguish among them.

We say that a state $\rho \in \Omega$ is *broadcast* by an affine mapping $\phi : \Omega \rightarrow \Omega \otimes \Omega$ iff the bipartite state $\phi(\rho)$ has marginal states $\phi(\rho)_1$ and $\phi(\rho)_2$ both equal to ρ . If ρ can be expressed as a mixture of distinguishable — hence, clonable — states $\alpha_1, \dots, \alpha_n$, say $\rho = \sum_i t_i \alpha_i$, then one can broadcast ρ using a cloning map ϕ for the states $\alpha_1, \dots, \alpha_n$: the state $\phi(\rho) = \sum_i t_i \alpha_i \otimes \alpha_i$ has both marginal states equal to ρ , as required. The quantum *no-broadcasting theorem* of Barnum et al. [8] tells us that, conversely, two *quantum* states are jointly broadcastable iff, regarded as density operators, they commute — which, by the Spectral Theorem, is equivalent to requiring that all are convex combinations of some single set of distinguishable pure states. In fact, this is a corollary of a more general result:

Theorem 7 ([9, 10]). *Let Γ be the set of states broadcast by an affine mapping $\phi : \Omega \rightarrow \Omega \otimes \Omega$. Then Γ is the simplex generated by a set of distinguishable states in Ω , which are cloned by ϕ .*

(Although we omit the proof here, it is not especially difficult. This is in contrast to earlier proofs of the quantum no-broadcasting result [8, 45], which were not especially easy.)

4.2 Remote Evaluation

Suppose \mathcal{C} is a locally tomographic, monoidal probabilistic theory. Consider two parties, Alice and Bob, occupying arbitrarily distant sites. Suppose that Alice controls a pair of systems, say $A_o, A_1 \in \mathcal{C}$, while Bob controls a system $B \in \mathcal{C}$. Since \mathcal{C} is monoidal, we can represent Alice's two systems together as a single bipartite system $A = A_o A_1$, and the entire Alice-Bob system, by the tripartite composite $AB = (A_o A_1) B \simeq A_o (A_1 B)$.

Now suppose that the composite system A_1B is in a state ω , while Alice's system A_o is in a state α , independent of the A_1B sub-system. Then the total state of the system $AB = A_o(A_1B)$ is $\alpha \otimes \omega$. Now let Alice make a measurement on her system $A = A_oA_1$, obtaining a result represented by an effect $f \in \mathbf{E}(A)$; suppose Bob also makes a measurement on his system, B , obtaining a result represented by an effect $b \in \mathbf{E}(B)$, so that the joint outcome of these two measurements is $f \otimes b$.

Lemma 5 (Remote Evaluation). *With notation as above, let $\widehat{\omega} : \mathbf{E}(A_1) \rightarrow \mathbf{V}(B)$ and $\widehat{f} : \mathbf{V}(A_o) \rightarrow \mathbf{E}(A_1)$ be the conditioning and co-conditioning maps associated with the state ω and the effect f . Then, for all $\alpha \in \mathbf{V}(A_o)$ and all $b \in \mathbf{E}(B)$,*

$$(\alpha \otimes \omega)(f \otimes b) = \widehat{f}(\widehat{\omega}(\alpha))(b). \quad (2)$$

The proof is easy: one simply checks that the formula is correct when ω is a product state and f is a product effect. Since we are working with locally tomographic composites, product states and effects span $\mathbf{E}(A_1B)^*$ and $\mathbf{E}(A_oA_1)$, respectively, so (2) holds for all choices of ω and f . Nevertheless, the result is somewhat surprising, for it asserts that the mapping

$$\tau := \widehat{\omega} \circ \widehat{f} : \mathbf{V}(A_o) \rightarrow \mathbf{V}(B)$$

can be implemented, probabilistically, by means of a preparation of A_1B in the joint state ω and a (successful) observation of f on A_oA_1 . In particular, when Alice observes the effect f , the corresponding un-normalized conditional state of Bob's system is

$$(\alpha \otimes \omega)(f \otimes -) = \tau(\alpha).$$

Note that the probability of the process τ occurring in state α is $u_B(\tau(\alpha))$, which is exactly the marginal probability $(\alpha \otimes \omega)_1(f)$ of Alice's obtaining f . In what follows, we refer to the pair (f, ω) as a *remote evaluation* protocol for the process $\tau = \widehat{f} \circ \widehat{\omega}$.

We can reformulate the notion of conditioning and co-conditioning map, and the remote evaluation Lemma (Lemma 5), in purely categorical terms. In fact, both make sense in any symmetric monoidal category \mathcal{C} . Given objects $A, B \in \mathcal{C}$ and a morphism $\omega : A \otimes B \rightarrow I$, there is a canonical mapping $\widehat{\omega} : \mathcal{C}(I, A) \rightarrow \mathcal{C}(B, I)$ given by

$$\begin{array}{ccc} B & \xrightarrow{\alpha \otimes \text{id}_B} & A \otimes B \\ & \searrow \widehat{\omega}(a) & \downarrow \omega \\ & & I \end{array} \quad (3)$$

Dually, if $f \in \mathcal{C}(I, A \otimes B)$, there is a natural mapping $\widehat{f} : \mathcal{C}(A, I) \rightarrow \mathcal{C}(I, B)$ given by

$$\begin{array}{ccc} I & \xrightarrow{f} & A \otimes B \\ & \searrow \widehat{f}(\alpha) & \downarrow \alpha \otimes \text{id}_B \\ & & B \end{array} \quad (4)$$

If \mathcal{C} is a monoidal probabilistic theory, then $\widehat{\omega}$ and \widehat{f} , defined in this way, correspond exactly to the conditioning and co-conditioning maps associated with the bipartite state $\omega : A \otimes B \rightarrow I$ and effect $f : I \rightarrow A \otimes B$. Combining diagrams (3) and (4), and taking advantage of the monoidal structure of \mathcal{C} — in particular, the fact that $\alpha \otimes \omega = (I \otimes \omega) \circ (\alpha \otimes \text{id}_{A \otimes B})$ — we have

$$\widehat{\omega}(\widehat{f}(\alpha) \otimes \text{id}_B) = \omega \circ (\alpha \otimes \text{id}_{A_1B}) \circ (f \otimes \text{id}_B) = (\alpha \otimes \omega) \circ (f \otimes \text{id}_B) \quad (5)$$

which precisely expresses Lemma 5.

$$\begin{array}{ccc}
 & & A_o \otimes A_1 \otimes B \\
 & \nearrow f \otimes B & \downarrow \alpha \otimes \text{id}_{A_1 B} \\
 B = I \otimes B & \xrightarrow{\widehat{f}(\alpha) \otimes \text{id}_B} & A_1 \otimes B \\
 & \searrow \widehat{\omega}(\widehat{f}(\alpha)) & \downarrow \omega \\
 & & I
 \end{array} \tag{6}$$

This has an important corollary. Since $\omega \circ (\alpha \otimes \text{id}_{A_1 B}) = \alpha \otimes (\text{id}_{A_o} \circ \omega)$, we can re-write (6) as

$$\widehat{\omega}(\widehat{f}(\alpha) \otimes \text{id}_B) = \alpha \circ (\text{id}_{A_o} \otimes \omega) \circ (f \otimes \text{id}_B)$$

Thus, the dual process $\tau^* : \mathbf{E}(B) \rightarrow \mathbf{E}(A)$ corresponding to the process $\tau = \widehat{\omega} \circ \widehat{f}$ arising in the remote evaluation protocol, is in fact a morphism in $\mathcal{C}(A_o, B)$.

Conclusive Teleportation In the special case in which the models A_o, A_1 and B are isomorphic and weakly self-dual, we can consider a remote evaluation protocol in which both the effect $f \in \mathbf{E}(A)$ and the state $\omega \in \Omega(A_1 B)$ correspond to order isomorphisms $\widehat{f} : \mathbf{V}(E_o) \simeq \mathbf{E}(A_1)$ and $\widehat{\omega} : \mathbf{E}(A_1) \simeq \mathbf{V}(B)$. In this case, the process $\tau = \widehat{\omega} \circ \widehat{f}$ is again an order-isomorphism. If this scenario is repeated many times, Bob can perform sufficiently many measurements to determine $\tau(\alpha)$ with reasonable confidence, and then *compute* the value of α . On the other hand, if τ is probabilistically reversible, in a single run of the scenario Bob can actually correct his state, with non-zero probability, so that it agrees with α . In this case, we may say that the state α has been *teleported* from Alice's system A_o to Bob's system B , and refer to (f, ω) as a *teleportation protocol*. If τ is reversible with probability 1, we shall say that (f, ω) is a *strong* teleportation protocol.

Deterministic Teleportation Suppose now that Alice has access to an observable $\{f_i\}$ on $A = A_o A_1$, with each of the effects f_i an isomorphism effect. Each of these effects, in combination with the isomorphism state ω , gives rise to a conclusive teleportation protocol, implementing the order-isomorphism $\tau_i = \widehat{\omega} \circ \widehat{f}_i : \mathbf{V}(A_o) \simeq \mathcal{G}(B)$. If Alice is permitted to communicate (classically) with Bob, then upon observing outcome f_i , she can instruct Bob to implement the inverse process τ_i^{-1} , which he can do with probability $c_i := u_B \tau_i^{-1}(\alpha)$. It follows that the post-measurement state of Bob's system will be $\sum_i c_i \alpha = \alpha$. particular, $\sum_i c_i = \sum_i u_B \tau_i^{-1}(\alpha)$. Say that A *supports a deterministic teleportation protocol* iff there exists such an observable $\{f_i\}$ and such a state ω .

Theorem 8 ([11]). *Suppose there exist a finite group G acting transitively on A 's pure states, and a G -equivariant order-isomorphism $\mathbf{E}(A) \simeq \mathbf{E}(A)^*$. Then A supports a deterministic teleportation protocol.*

Entanglement Swapping Suppose that, like Alice, Bob controls a bipartite system $B = B_1 B_2$. Assume here that A_o, A_1, B_1 and B_0 are all isomorphic to one another. Given an entangled state ω between A_1 and B_1 , and isomorphism effects f on $A = A_o A_1$ and g on $B = B_1 B_0$, we find that, for any state μ on $A_o A_0$, we have (up to the obvious symmetrizers and associators)

$$(f \otimes g)(\mu \otimes \omega) = g(\widehat{\omega} \circ \widehat{f} \circ \widehat{\mu}^*).$$

Since this holds for any choice of $g \in \mathbf{E}(B)$, we have

$$(\mu \otimes \omega)_{B|f} = \widehat{\omega} \circ \widehat{f} \circ \widehat{\mu}^*$$

If $\tau = \widehat{\omega} \circ \widehat{f}$ is probabilistically reversible, then upon Bob's executing the reverse process, the state μ has been transferred from $A_o B_2$ to $B = B_1 B_0$.

Teleportation and Compact Closure Let \mathcal{C} be any symmetric monoidal category. A *dual* for an object $A \in \mathcal{C}$ is an object $B \in \mathcal{C}$, together with two morphisms, $\eta : I \rightarrow B \otimes A$ and $\epsilon : A \otimes B \rightarrow I$ — called the *unit* and *co-unit*, respectively — such that

$$(\epsilon \otimes \text{id}_A) \circ (\text{id}_A \otimes \eta) = \text{id}_A \quad \text{and} \quad (\text{id}_B \otimes \epsilon) \circ (\eta \otimes \text{id}_A) = \text{id}_B \quad (7)$$

In view of the discussion above, if \mathcal{C} is a monoidal probabilistic theory and f, ω is a conclusive teleportation protocol for a pair of systems $A, B \in \mathcal{C}$, then the remote evaluation lemma tells us that f and ω function as a unit and co-unit, respectively, for A and B . A symmetric monoidal category in which every object has a dual is said to be *compact closed*. A *compact structure* on a compact closed category is a specification, for every object $A \in \mathcal{C}$, of a distinguished dual $A' \in \mathcal{C}$. Where $A = A'$ for every $A \in \mathcal{C}$, this structure is *degenerate*.¹⁴

Proposition 9 ([15]). *Let \mathcal{C} be a monoidal probabilistic theory. The following are equivalent.*

- (a) \mathcal{C} admits a compact closed structure.
- (b) Every $A \in \mathcal{C}$ can be teleported through some $B \in \mathcal{C}$;
- (c) Every morphism in \mathcal{C} has the form $\widehat{\omega} \circ \widehat{f}$ for some bipartite state ω and bipartite effect f in \mathcal{C} .

Proof: The equivalence of (a) and (b) is clear from the preceding discussion. To see that these are in turn equivalent to (c), suppose first that (a) and (b) hold. Choose for each $A \in \mathcal{C}$ a dual system A' , a state $\omega_A \in \mathcal{C}(A \otimes A', I)$, and an effect $f_A \in \mathcal{C}(I, A' \otimes A)$ with $\widehat{\omega}_A = \widehat{f}_A^{-1}$. Then for any morphism $\tau \in \mathcal{C}(A, B)$, let $f_\tau \in \mathcal{C}(I, A \otimes B)$ be the effect $f_A \circ (A' \otimes \tau)$. It is easily checked that then $\widehat{f}_\tau = \tau \circ \widehat{f}_A$, so that $\tau = \widehat{f}_\tau \circ \widehat{\omega}_A$. Conversely, if (c) holds, then for each A , the identity mapping id_A factors as $\widehat{\omega}_A \circ \widehat{f}_A$ for some $\omega_A \in \mathcal{C}(B \otimes A, I)$ and some $f \in A \otimes B$. It follows that $\widehat{\omega}_A = \widehat{f}_A^{-1}$, so this gives us a compact closed structure. \square

4.3 Steering

Let B be a probabilistic model. An *ensemble* for a state $\beta \in \Omega(B)$ is a finite set of states $\beta_i \in \mathbf{V}(B)_+$ such that $\sum_i \beta_i = \beta$. We can understand such an ensemble as representing one possible way of *preparing* the state β , namely, to choose one of the normalized states $\widehat{\beta}_i := \beta_i / u(\beta_i)$ with probability $p_i = u_B(\beta_i)$.

One way to do *this* is to begin with a bipartite state ω on a non-signaling composite AB , with marginal $\omega_2 = \beta$. Then for any observable $E = \{a_i\}$ on A , the un-normalized conditional states $\beta_i := \widehat{\omega}(a_i)$ are an ensemble for β . That is: by measuring E , we prepare not only the marginal state ω_B , but a *particular ensemble* for this state. By choosing to measure a different observable, we will typically obtain a different ensemble for β . If A and B are *quantum* systems, and if ω is a pure entangled state of AB , then *any* ensemble for ω_2 can be obtained in this way from a suitable choice of measurement on A . This phenomenon was first observed by Schrödinger [58], who called it *steering*. The concept extends readily to the setting of an arbitrary non-signaling composite.

Definition 18. Let AB be a non-signaling composite of probabilistic models A and B . A bipartite state $\omega \in AB$ is *steering for its B marginal*, or *B -steering*, for short, iff, for every ensemble (convex decomposition) $\omega_2 = \sum_i \beta_i$, where β_i are un-normalized states of B , there exists an observable $E = \{a_i\}$ on A with $\beta_i = \widehat{\omega}(a_i)$. We say that ω is *bi-steering* iff it's steering for both marginals.

The relevance of steering to information processing became evident when Bennett and Brassard [?], in the same paper that introduced quantum key distribution, considered a natural quantum scheme for another important cryptographic primitive, bit commitment, and showed that ensemble

¹⁴Duals, where they exist, are canonically isomorphic. Hence, for most purposes, the choice of one rather than another object as “the” dual is irrelevant. The existence of a degenerate compact structure is, however, a real constraint [15, 60].

steering can be used to break it. In the proposed scheme, the two possible values to which Alice can commit are represented by two distinct ensembles for the same density matrix. She is to send samples from the ensemble to Bob in order to commit, and later reveal which states she drew so that Bob can check that she used the claimed ensemble. However, by sending to Bob, not a draw from the ensemble, but one of two systems in an entangled pure bipartite state with the specified density matrix as its marginal. Keeping the other system, she can realize either ensemble after she has already sent the systems to Bob by making measurements on her entangled system, enabling her to perfectly mimic commitment to either bit.

Later Mayers, and Lo and Chau, showed that *no* information-theoretically secure quantum bit commitment protocol can exist. The techniques they used to defeat putative protocols do not literally use steering, but are closely related to the Bennett-Brassard steering attack, in particular in Alice's retention of a system *purifying* the systems she sends to Bob in the course of the protocol.

The paper [14] studies steering in the context of general probabilistic theories. If α is any state on A and β is a *pure* state on B , then $\omega = \alpha \otimes \beta$ is trivially steering for $\omega_2 = \beta$ since the latter has no non-trivial ensembles. In particular, any pure product state will be steering for both of its marginals. Any isomorphism state $\omega \in \mathbf{V}(AB)$ will also be steering.

It follows almost immediately from the definition, that if ω is steering for its B -marginal, then the image, $\widehat{\omega}(\mathbf{E}(A)_+)$, of the positive cone in $\mathbf{E}(A)$, is a face of $\mathbf{V}(B)_+$. Indeed, we have

Lemma 6. *If ω is steering, then $\widehat{\omega}(\mathbf{E}(A)_+) = \text{Face}(\omega_2)$.*

Here $\text{Face}(\omega_2)$ refers to the face generated by ω_2 , i.e, the smallest face of $\mathbf{V}(B)_+$ containing ω_2 . The converse of Lemma (6) is false.

A probabilistic theory \mathcal{C} *supports uniform universal steering* if, for every system $B \in \mathcal{C}$, there exists a system $A_B \in \mathcal{C}$ such that every state $\beta \in A$ is the marginal of some B -steering state $\omega \in A_B B$. If one can always take $A_B = A$, we say that \mathcal{C} *supports universal self-steering*.

Proposition 10. *Let $\omega \in \Omega(AB)$ be steering for ω_2 , where ω_2 is interior to $\mathbf{V}(B)_+$, so that $\text{Face}(\omega_2) = \mathbf{V}(B)_+$. If $\widehat{\omega}$ is injective (non-singular), then $\widehat{\omega}$ is an order isomorphism. If $\mathbf{V}(B)$ is irreducible, therefore, by Proposition 4, $\widehat{\omega}$ is pure.*

In other words, if A and B have the same dimension, then the states that are steering for an *interior marginal* are precisely the isomorphism states (and hence, are steering for both marginals).

Steering is closely related to an important property of quantum theory called *homogeneity*.

Definition 19. Let \mathcal{G} be a group of order-automorphisms of an ordered vector space \mathbf{E} . We say that \mathbf{E} is *homogeneous with respect to \mathcal{G}* if \mathcal{G} acts transitively on the *interior* of the positive cone \mathbf{E}_+ . That is, for every pair of interior points a, b of \mathbf{E}_+ , there exists an element $g \in \mathcal{G}$ with $ga = b$. We say \mathbf{E} is *homogeneous* if it is homogeneous with respect to some group of order-automorphisms, or, equivalently, if it is homogeneous with respect to the group $\text{Aut}(\mathbf{E})$ of all order-automorphisms.

It can be shown that the cone $L_+(\mathcal{H})$ of positive operators on a finite-dimensional Hilbert space \mathcal{H} is homogeneous with respect to the group of order-automorphisms of $L(\mathcal{H})$. As we discuss below in Section 5, the combination of homogeneity and strong self-duality comes close to characterizing finite-dimensional quantum theory among probabilistic theories generally. More precisely, the *Koecher-Vinberg Theorem* asserts that if \mathbf{E} is an ordered linear space whose positive cone \mathbf{E}_+ is both homogeneous and self-dual, then \mathbf{E} can be given the structure of a euclidean Jordan algebra. With this in mind, the following result is particularly intriguing:

Theorem 11. *For a model with irreducible state space $\mathbf{V}(A)$ the following are equivalent:*

- (a) *A is homogeneous;*
- (b) *Every normalized state in the interior of $\Omega(A)$ is the A -marginal of an isomorphism state in $B \otimes_{\max} A$, where B is any (fixed) model with state space order-isomorphic to $\mathbf{V}(A)^*$.*

From this we obtain:

Corollary 12. *For any model with irreducible state space A , the following are equivalent:*

- (a) *$\mathbf{V}(A)_+$ is weakly self-dual and homogeneous;*

(b) Every normalized state in the interior of $\Omega(A)$ is the marginal of an isomorphism state in $A \otimes_{\max} A$.

Corollary 10, combined with Theorem 11, gives

Proposition 13. *In any theory that supports universal uniform steering, every irreducible, finite-dimensional state space in the theory is homogeneous.*

In light of Corollary 12, we also have

Proposition 14. *In any theory that supports universal self-steering, every irreducible, finite-dimensional state space in the theory is homogeneous and weakly self-dual.*

Therefore, the distance between probabilistic theories allowing universal self-steering, and those whose state-spaces are Jordan-algebraic is just that between weak and strong self-duality.

In [13] it was shown that an asymptotically exponentially secure bit commitment protocol, based (like the original Bennett-Brassard one-qubit protocol) on the nonuniqueness of convex decomposition in nonclassical state spaces, exists in any theory containing some nonclassical state spaces, coupled only by the minimal tensor product (so that there is no entanglement between them). In a nonclassical theory in which all states can be steered, by contrast, this type of bit commitment protocol can always be defeated.

4.4 Entropy and Information Causality

Classical information theory begins with the Gibbs-Shannon entropy $H(p) = -\sum_i p_i \log(p_i)$ of a discrete probability weight p_1, \dots, p_n . Analogously, in quantum theory the *von Neumann entropy* of the state corresponding to a density operator ρ is given by $S(\rho) := \text{Tr} \rho \log \rho$. This is related to the classical Gibbs-Shannon entropy in two important ways. On one hand, $S(\rho)$ is the minimum of the Gibbs-Shannon entropies $-\sum_i p_i \log p_i$ of the probability weights $p_i = \text{Tr}(\rho e_i)$ that ρ induces on quantum tests $\{e_i\}$. (This turns out to be achieved when the measurement is in a diagonalizing basis). Alternatively, $S(\rho)$ is the minimum Gibbs-Shannon entropy of the probabilities p_i arising in representations of ρ as a mixture $\rho = \sum_i p_i \rho_i$ of pure states ρ_i . (This again turns out to be achieved for an ensemble whose states are the rank-one projectors corresponding to a diagonalizing basis).

Both of these characterizations make sense in the context of an arbitrary probabilistic model, but in general, they are not equivalent.

Definition 20. *Let α be a state on A . For each test $E \in \mathbf{M}(A)$, define the local measurement entropy of α at E , $H_E(\alpha)$, to be the classical (Shannon) entropy of $\alpha|_E$, i.e.,*

$$H_E(\alpha) := - \sum_{x \in E} \alpha(x) \log(\alpha(x)).$$

The measurement entropy of α , $H(\alpha)$, is the infimum of $H_E(\alpha)$ as E ranges over $\mathbf{M}(A)$, i.e.,

$$H(\alpha) := \inf_{E \in \mathbf{M}(A)} H_E(\alpha).$$

Note that the measurement entropy of a state $\alpha \in \Omega(A)$ depends entirely on the structure of the test space $\mathbf{M}(A)$, and not on the geometry of the state space Ω .

We shall assume in what follows that the measurement entropy of a state is actually achieved on some test, i.e., that $H(\alpha) = H_E(\alpha)$ for some $E \in \mathbf{M}(A)$. This is the case in quantum theory, and can be shown to hold much more generally, given some rather weak analytic requirements on the model A ([12], Appendix B.) It follows that $H(\alpha) = 0$ if and only if there is a test such that α assigns probability 1 to one of its outcomes.

Notation: It will often be convenient to write $H(\alpha)$ as $H(A)$, where context makes clear which state is being considered. If AB is a non-signaling composite, and $H(AB)$ represents $H(\omega)$, we shall

write $H(A)$ and $H(B)$ for the marginal entropies $H(\omega_1)$ and $H(\omega_2)$. It is easily checked that the measurement entropy is *subadditive*, i.e.,

$$H(AB) \leq H(A) + H(B).$$

Definition 21. Let α be a state on A . The mixing (or preparation) entropy for α , denoted $S(\alpha)$, is the infimum of the classical (Shannon) entropy $H(p_1, \dots, p_n)$ over all finite convex decompositions $\alpha = \sum_i p_i \alpha_i$ with α_i pure states in $\Omega(A)$.

Again, we write $S(A)$ for $S(\alpha)$ where α belongs to the state space Ω of a system $A = (\mathfrak{M}, \Omega)$. In contrast to measurement entropy, the mixing entropy of a state depends only on the geometry of the state space Ω , and is independent of the choice of test space $\mathfrak{M}(A)$. The mixing entropy is essentially the same as the entropy defined for elements of compact convex sets by A. Uhlmann in [?].

We call a theory *monoentropic* if mixing entropy equals measurement entropy, for every state of every model in the theory. Appendix B of [12] considers some implications of monoentropicity. For instance, it is shown that any monoentropic model A in which the set of pure states is closed in $\Omega(A)$ is sharp.

We define conditional and mutual information in terms of measurement entropy via formulas that also hold classically:

Definition 22. The conditional measurement entropy between A and B is defined to be

$$H(A|B) := H(AB) - H(B). \quad (8)$$

The [measurement-based] mutual information is defined to be:

$$I(A : B) := H(A) + H(B) - H(AB). \quad (9)$$

Intuitively, one might expect that $I(A : B)$ should not *decrease* if we recognize that B is a part of some larger composite system BC – i.e., we might expect that $I(A : B) \leq I(A : BC)$. Simple algebraic manipulations (using Eqs. (8) and (9)) allow us to reformulate this condition in various ways.

Lemma 7. The following are equivalent:

- (a) $I(A : BC) \geq I(A : B)$
- (b) $H(A|BC) \leq H(A|B)$
- (c) $H(AB) + H(BC) - H(B) \leq H(ABC)$
- (d) $I(A : B|C) \geq 0$, where $I(A : B|C) = H(A|C) + H(B|C) - H(AB|C)$.

The measurement entropy is said to be *strongly subadditive* if it satisfies the equivalent conditions (a)-(d). (Condition (c) is what is usually termed “strong subadditivity” (SSA).) A probabilistic theory in which conditions (a)-(d) are satisfied for all systems A, B and C will also be called *strongly subadditive*. Despite the intuition mentioned above, strong subadditivity can fail in general theories, which is perhaps a signal that mutual information as defined above should not be interpreted in general as “the information each system contains about the other”.

The Holevo Bound and the Data Processing Inequality The strong subadditivity inequality is crucial to deriving bounds on many quantum information-transmission protocols, and the conditions under which it is satisfied with equality are also of great importance. Another extremely important inequality – derivable, in the quantum setting, from strong subadditivity – is the *Holevo bound*, which figures in an expression for the highest achievable rate of classical information transmission through a noisy quantum channel.

The standard formulation of the Holevo bound can apply to a general theory, if the entropies are interpreted as measurement entropies: it asserts that if Alice prepares a state $\rho = \sum_{x \in E} p_x \rho_x$ for Bob, then, for any measurement F that Bob can make on his system,

$$I(E : F) \leq \chi,$$

where $\chi := H(\rho) - \sum_{x \in E} p_x H(\rho_x)$ (often called the *Holevo quantity*).

Suppose that Alice has a classical system $A = (\{E\}, \Delta(E))$ and Bob a general system B . Alice's system is to serve as a record of which state of B she prepared. The situation above is modeled by the joint state $\omega^{AB} = \sum_{x \in E} p_x \delta_x \otimes \beta_x$, where δ_x is a deterministic state of Alice's system with $\delta_x(x) = 1$. Bob's marginal state is $\omega_B = \sum_{x \in E} p_x \beta_x$. By Lemma ??, $H(\omega^{AB}) = H(A) + \sum_{x \in E} p_x H(\beta_x)$. Hence,

$$\begin{aligned} I(A : B) &= H(A) + H(B) - H(AB) \\ &= H(A) + H(B) - \left(H(A) + \sum_{x \in E} p_x H(\beta_x) \right) \\ &= H(\omega_B) - \sum_{x \in E} p_x H(\beta_x) = \chi. \end{aligned}$$

So the content of the Holevo bound is simply that the mutual information between the measurement of Alice's classical system and any measurement on Bob's system is no greater than $I(A : B)$,

$$I(E : F) \leq I(A : B).$$

(While this is certainly natural, in general theories it does not always hold.)

Both strong subadditivity and the Holevo bound are instances of a more basic principle. The *data processing inequality* (DPI) asserts that, for any systems A , B and C , and any physical process $\mathcal{E} : B \rightarrow C$,

$$I(A : \mathcal{E}(B)) \leq I(A : B)$$

where $I(A : \mathcal{E}(B))$ refers to the mutual information of the state resulting from applying $\text{id}_A \otimes \mathcal{E}$ to the state of AB . The strong subadditivity of entropy amounts to the DPI for the process that simply discards a system (the *marginalization map* $BC \rightarrow C$). The Holevo bound is the DPI for the special case of measurements, which can be understood as processes taking a system into a classical system which records the outcome.

Information Causality In a widely discussed paper [53], M. Pawłowski *et al.* introduced a constraint on a non-signaling probabilistic theory, which they called *information causality*, in terms of the following protocol. Two parties, Alice and Bob, share a joint non-signaling state, known to both of them. Alice receives a random bit string e of length N ; after making measurements, she sends Bob message, f , a bit-string of length m or less. Bob receives a random variable G , encoding a number, $k = 1, \dots, N$, which he takes as the instruction to measure Alice's k -th bit. After making a suitable measurement, and taking into account both its outcome and Alice's message, Bob produces his guess, b_k . Information causality is the requirement that

$$\sum_{k=1}^N I(e_k : b_k | G = k) \leq m. \quad (10)$$

The main result of [53] is that if a theory contains states that violate the CHSH inequality by more than the Tsirel'son bound, then it violates information causality. In particular, if Alice and Bob can share PR boxes, then using a protocol due to van Dam [?], they can violate information causality maximally, meaning that Bob's guess is correct with certainty, and the left hand side of Equation (10) is N . Pawłowski *et al.* also give a proof, using fairly standard manipulations of quantum mutual information, that quantum theory *does* satisfy information causality.

One of the principle results of [12] is a sufficient condition for a general probabilistic theory to be information-causal. The following is a strengthening of that result:

Theorem 15. *Suppose that a theory is strongly subadditive, and satisfies the Holevo bound. Then the theory satisfies information causality. It follows that any theory satisfying these conditions cannot violate Tsirel'son's bound.*

Since strong subadditivity and the Holevo bound follow from the data processing inequality, we have the following:

Corollary 16. *Any theory in which measurement-based mutual information satisfies the data processing inequality satisfies information causality.*

In [12], monoentropicity was assumed in addition to SSA and Holevo. As noted there, it was only used to derive that $H(A|B) \geq 0$ when A is classical. However, this follows easily from strong subadditivity in the equivalent (cf. Lemma 7) form $I(A : B|C) \geq 0$, when we let A and B be identical perfectly correlated classical systems. We have

$$I(A : B|C) = H(A|C) + H(B|C) - H(AB|C) \quad (11)$$

$$= H(AC) - H(C) + H(BC) - H(C) - H(ABC) + H(C) \quad (12)$$

$$= H(AC) + H(BC) - H(ABC) - H(C). \quad (13)$$

$$(14)$$

Since A, B are perfectly correlated classical systems, $H(AC) = H(BC) = H(ABC)$. Consequently, in this case $I(A : B|C) = H(AC) - H(C) \equiv H(A|C)$. By SSA, this is ≥ 0 .¹⁵

4.5 Other developments

There is much more to say about information processing in general probabilistic theories than we have room to discuss here. We remark in particular on [20], in which a version of the deFinetti theorem is proved for states on test spaces.

5 Characterizing Quantum Theory

As we've seen, a great number of information-processing phenomena first discovered in association with quantum theory, are actually rather more generally *post-classical*, rather than specifically quantum-mechanical, in character. This brings us back to the question of how to *characterize* quantum theory in operational or probabilistic terms. The idea is to identify one or more features of quantum theory that can be expressed in purely operational-probabilistic terms — roughly, without any special reference to the Hilbert space structure, but only in terms of primitive concepts such as states, effects, tests, processes, etc. — and that, taken together, *uniquely* specify quantum (or quantum-plus-classical) models. This is an old problem, and also a somewhat vague one, since what counts as a satisfactory solution will be, to some extent, a matter of taste. Even so, striking progress has been made in the past several years, leading to several different, more-or-less satisfactory characterizations of quantum mechanics as a probability theory [refs]. have been found. In this section, we review one of these [17, 70, 72, 18], which makes use of the equivalence between homogeneous self-dual cones and Euclidean Jordan algebras.

5.1 Homogeneity and Self-Duality

Let \mathbf{E} be (for the moment) any finite-dimensional ordered linear space. Given a bilinear form $\mathcal{B} : \mathbf{E} \times \mathbf{E} \rightarrow \mathbb{R}$, we define the *internal dual* (with respect to \mathcal{B}) of the cone \mathbf{E}_+ to be the cone

$$\mathbf{E}^+ := \{a \in \mathbf{E} | \forall x \in \mathbf{E}_+, \mathcal{B}(a, x) \geq 0\}.$$

We say that \mathcal{B} is *positive on \mathbf{E}_+* , or simply *positive*, iff $\mathbf{E}_+ \subseteq \mathbf{E}^+$ — in other words, if the linear mapping $\beta : \mathbf{E} \rightarrow \mathbf{E}^*$ given by $\beta(a)(x) = \mathcal{B}(a, x)$ is positive.

¹⁵ The realization that Theorem 4 of [12] could be strengthened this way grew out of discussions between some of the authors of [12] while the article was in press, but too late for inclusion in the published version.

Definition 23. \mathbf{E} is *self-dual with respect to* \mathcal{B} iff $\mathbf{E}^+ = \mathbf{E}_+$. We shall say that \mathbf{E} is *weakly self-dual* iff there exists a bilinear form \mathcal{B} with respect to which \mathbf{E} is self-dual, and *strongly self-dual*, if there exists an *inner product* on \mathbf{E} having this feature.

Weak self-duality is equivalent to the existence of an isomorphism state in $A \otimes_{\max} A$. As discussed above, this is equivalent to the requirement that there exist *some* composite of three copies of A that supports a teleportation protocol, and to the requirement that states on A arise as marginals of steering states in a composite of A with itself [14]. Strong self-duality is much less easy to motivate, but we will discuss several ways in which it can be justified in the next section.

Recall that \mathbf{E} is *homogeneous* with respect to a group \mathcal{G} of order-automorphisms if \mathcal{G} acts transitively on the *interior* of the positive cone \mathbf{E}_+ , so that for every pair of interior points a, b of \mathbf{E}_+ , there exists an element $g \in \mathcal{G}$ with $ga = b$.

Classical and quantum probabilistic models are both homogeneous and self-dual. Somewhat more generally, let \mathbf{E} be a euclidean Jordan algebra. This is a finite-dimensional real vector space \mathbf{E} equipped with a commutative bilinear operation \bullet satisfying the *Jordan identity* $a^2 \bullet (b \bullet a) = (a^2 \bullet b) \bullet a$ for all $a, b \in \mathbf{E}$, and equipped with a canonical trace such that $\langle a, b \rangle := \text{Tr}(a \bullet b)$ is an inner product, with $\langle a \bullet b, c \rangle = \langle a, b \bullet c \rangle$ for all $a, b, c \in \mathbf{E}$. The set $\mathbf{E}_+ = \{a^2 | a \in \mathbf{E}\}$ (where $a^2 = a \bullet a$) is a cone in \mathbf{E}_+ , and one can show is homogeneous with respect to the group of order-automorphisms of \mathbf{E} , and self-dual with respect to the tracial inner product. Remarkably, there is a converse, to be found in work of M. Koecher [42] and E. Vinberg [65]

If G be any closed subgroup of $\text{Aut}(\mathbf{E})$, acting transitively on the interior of \mathbf{E}_+ , then G is a Lie subgroup of $GL(\mathbf{E})$. Let \mathfrak{g} denote its Lie algebra, and let \mathfrak{g}_u denote the Lie algebra of the stabilizer $G_u \leq G$ of the order-unit. The following formulation of the Koecher-Vinberg Theorem summarizes the construction of the Jordan product on \mathbf{E} . See [30] for a proof (also, the Appendix to [18] contains a fairly detailed outline of the proof and some additional remarks pertinent to the precise version given above):

Theorem 17 (Koecher-Vinberg). *Let \mathbf{E}_+ be self-dual with respect to some inner product on \mathbf{E} , and let G be a closed, connected subgroup of $\text{Aut}(\mathbf{E})$, acting transitively on the interior of \mathbf{E}_+ . Then*

- (a) *It is possible to choose a self-dualizing inner product on \mathbf{E}_+ in such a way that $G_u = G \cap \mathcal{O}(\mathbf{E})$ (where $\mathcal{O}(\mathbf{E})$ is the orthogonal group with respect to the inner product);*
- (b) *If $G = G^\dagger$ with respect to this inner product, then $\mathfrak{g}_u = \{X \in \mathfrak{g} | X^\dagger = -X\} = \{X \in \mathfrak{g} | Xu = 0\}$, and $\mathfrak{g} = \mathfrak{g}_u \oplus \mathfrak{p}$, where $\mathfrak{p} = \{X \in \mathfrak{g} | X^\dagger = X\}$;*
- (c) *In this case the mapping $\mathfrak{p} \rightarrow \mathbf{E}$, given by $X \mapsto Xu$, is an isomorphism. Letting L_a be the unique element of \mathfrak{p} with $L_a u = a$, define*

$$a \bullet b = L_a b$$

for all $a, b \in \mathbf{E}$. Then \bullet makes \mathbf{E} a formally real Jordan algebra, with identity element u .

In [41], Jordan, von Neumann and Wigner classified Euclidean Jordan algebras as belonging to one of two broad types, plus one exceptional example. These are

- (a) **Hermitian parts of matrix algebras** over \mathbb{R}, \mathbb{C} or \mathbb{H} , ordered as usual;
- (b) **Spin factors**, in which the normalized state space is a ball of dimension n ; and
- (c) **The Exceptional Jordan Algebra** of positive 2×2 hermitian matrices over the Octonions.

Thus, it would seem that if we can motivate both homogeneity and self-duality in operational terms, we will go a great way towards obtaining an operational characterization of finite-dimensional QM. This problem is taken up in the next section. We then discuss the consequences of assuming that a monoidal probabilistic theory consisting of Jordan models has locally tomographic composites. Here a theorem of H. Hanche-Olsen [38] can be invoked to show that, so long as the theory contains even a single instance of the simplest quantum-mechanical system — a qubit — every system allowed by the theory must be the theory must be quantum.

5.2 Motivating Homogeneity and Self-Duality

Let us call a model A *HSD* (Homogeneous and self-dual) iff its linear hull $\mathbf{E}(A)$ — or, equivalently, its dual, $\mathbf{V}(A)$ — is homogeneous and self-dual. Why should this be the case? In this section, we discuss several possible answers.

Homogeneity A model A is *uniform* iff the state space Ω contains a *uniform state* μ , i.e., one taking constant values $1/n$ on all outcomes of $X(A)$. Of course, this implies that all tests in $\mathfrak{M}(A)$ have cardinality n . For uniform systems, homogeneity of $\mathbf{E}(A)$ has a straightforward, natural and physically reasonable interpretation: it asserts that every non-singular state should be preparable, by means of a probabilistically reversible transformation, from the uniformly (or maximally) mixed state.¹⁶ As noted above, homogeneity is also implied by either of the following conditions:

- (a) Every interior state is the marginal of an isomorphism state
- (b) Every state is the marginal of a steering state.

Yet another way of arriving at the homogeneity of $\mathbf{V}(A)$ can be found in [70].

Self-Duality Self-duality seems less clear-cut, but can be obtained as a consequence of certain symmetry assumptions. Perhaps the simplest and most dramatic is the following beautiful result due to M. Mueller and C. Ududec. Call two states $\alpha, \beta \in \Omega(A)$ *sharply distinguishable by effects* iff there exists an effect a such that $\alpha(a) = 1$ and $\beta(a) = 0$. Mueller and Ududec call a system *bit-symmetric* iff every such pair of states can be mapped to any other such pair by a symmetry of the state cone, that is, an affine symmetry of Ω . They then prove:

Theorem 18 ([50]). *If $\Omega(A)$ is bit-symmetric, then $\mathbf{V}(A)$ (and hence, $\mathbf{E}(A)$) is self-dual.*

It is worth noting that not every self-dual model is bit-symmetric. For instance, if Ω is a 2-dimensional regular $2n + 1$ -gon, then $\mathbf{V}(\Omega)$ is self-dual, but Ω is not bit-symmetric. Bit-symmetry is thus a very restrictive, yet very plausible, and operationally meaningful, constraint.

A more involved condition having a somewhat similar flavor, but dealing with the test space structure $X(A)$ rather than the pure states of A , is worth mentioning. Call A *bi-symmetric* iff it is 2-symmetric under $G(A)$ and if $G(A)$ acts transitively on pure states. As discussed in Section 2.2, it is quite easy to construct such models one at a time. Recall that A is *sharp* iff for every outcome x , there is a unique state α with $\alpha(x) = 1$.

Theorem 19 ([72]). *Let \mathcal{C} be a monoidal probabilistic theory in which every model is bi-symmetric. If $A \in \mathcal{C}$ is irreducible and sharp, then $\mathbf{E}(A)$ is self-dual.*

Another way of obtaining self-duality from bi-symmetry involves the notion of a conjugate system:

Definition 24. *A conjugate for a model A is a structure $(\overline{A}, \gamma_A, \eta_A)$, where \overline{A} is a model, $\gamma_A : A \rightarrow \overline{A}$ is an isomorphism, and η_A is a bipartite state (on some non-signaling composite) $A\overline{A}$ such that*

$$\eta_A(x, \gamma_A(x)) = 1/n$$

for every $x \in X(A)$. We'll call γ_A the conjugation map and η_A , the correlator for the given conjugate.

Example 10. Let $A = A(\mathfrak{H})$ be the quantum model associated with a complex Hilbert space \mathfrak{H} , and $\overline{A} = A(\overline{\mathfrak{H}})$ associated with the conjugate Hilbert space. Define a mapping $\gamma_A : X(\mathfrak{H}) \rightarrow X(\overline{\mathfrak{H}})$ by $\gamma_A : x \mapsto \overline{x}$ (strictly speaking, the identity map!). Then, as discussed in Section 3.3, $\eta_A(x, \gamma_A(y)) = |\langle \Psi, x \otimes y \rangle|^2 = \text{Tr}(P_\Psi P_{x \otimes y})$ is a correlator.

If A has a conjugate, then it has a conjugate for which the correlator η_A is symmetric, in the sense that $\eta(x, \gamma_A(y)) = \eta(y, \gamma_A(x))$, and invariant, in the sense that $\eta_A(gx, \gamma_A(gy)) = \eta(x, \gamma_A(y))$.

¹⁶One might raise the *aesthetic* objection that it is awkward to make special reference to the interior state. But it is difficult to see how this is any worse aesthetically than making special reference to, say, pure states.

Indeed, $\eta^T(x, \gamma_A(y)) := \eta(y, \gamma_A(x))$ is again a correlator; averaging η and η^T gives us a symmetric correlator. If η is symmetric, then for all symmetries $g \in G(A)$, $\eta^g(x, y) = \eta(gx, gy)$ is again a symmetric correlator; averaging over G yields an invariant symmetric correlator. Henceforth, we assume that correlators are symmetric and invariant. It follows that the bilinear form

$$\mathcal{B}(a, b) := \eta(a, \gamma_A(b))$$

is *orthogonalizing*, meaning that $\mathcal{B}(x, y) = 0$ for all $x \perp y$ in $X(A)$. For the following, see [72]:

Theorem 20. *Let A be irreducible, bi-symmetric, and have a conjugate $(\bar{A}, \gamma_A, \eta_A)$. Then (a) \mathcal{B} is an inner product on \mathbf{E} , and (b) A is self-dual with respect to \mathcal{B} iff η_A is an isomorphism state iff A is sharp.*

5.3 HSD and Jordan Models

Call a model A *HSD* (Homogeneous and self-dual) iff the cone \mathbf{E}_+ is homogeneous under *some* group $\mathcal{G}(A)$ of order-automorphisms, and self-dual with respect to *some* inner product. If A is an HSD model, then by the Koecher-Vinberg theorem, $\mathbf{E}(A)$ carries a unique euclidean Jordan structure with respect to which the order unit, u , is the identity and $\langle a, u \rangle = \text{Tr}(a)$.

An *idempotent* in a Jordan algebra \mathbf{E} is an element $e \in \mathbf{E}_+$ with $e^2 = e \bullet e = e$. Idempotents in the special Jordan algebra $\mathbf{L}_h(\mathcal{H})$ are precisely orthogonal projection operators. A *primitive* idempotent is an idempotent that is not a sum of other non-zero idempotents; thus, in the context of $\mathbf{L}_h(\mathcal{H})$, a primitive idempotent is a rank-one projection operator. Any Euclidean Jordan algebra \mathbf{E} carries a canonical trace functional, with $\text{Tr}(ab) = \langle a, b \rangle$, and one can show that $\text{Tr}(e) = 1$ for any primitive idempotent. A *Jordan frame* in a Euclidean Jordan algebra \mathbf{E} is a set e_1, \dots, e_n of primitive idempotents summing to u . The Spectral Theorem for Euclidean Jordan algebras asserts that every $a \in \mathbf{E}$ has a unique representation as a sum of the form $\sum_{e \in E} t_e e$ over a Jordan frame E , where $\{t_e | e \in E\}$ are non-negative real coefficients. It follows that the extremal elements of the cone \mathbf{E}_+ are exactly the primitive idempotents. The group of order-automorphisms of \mathbf{E} fixing the unit u acts transitively on the set of Jordan frames, so all Jordan frames have the same size, the *rank* of \mathbf{E} . (Indeed, regarding the set of Jordan frames as a test space, this group acts fully transitively, i.e., any permutation of a Jordan frame can be implemented by an order-automorphism of \mathbf{E} .)

Definition 25. *A probabilistic model A is uniform iff its test have a uniform cardinality n , and the uniformly mixed probability weight $\mu(x) \equiv 1/n$ belongs to $\Omega(A)$.*

If A is an HSD model, then every primitive idempotent e in $\mathbf{E}(A)$ defines a pure state, $\langle e |$, and this is the unique pure state assigning probability 1 to the effect corresponding to e . By a *Jordan model*, we mean an HSD model A such that every outcome in $X(A)$ is a primitive idempotent in $\mathbf{E}(A)$, or, equivalently, every test is a Jordan frame. Evidently, such a model is unital, indeed, sharp, and uniform.

There is a converse. Suppose A is HSD. By an easy extension of the converse to the Krein-Mil'man theorem, any closed, generating subset of $\mathbf{V}(A)_+$ contains every a point on every extremal ray of $\mathbf{V}(A)_+$. By our standing assumption of outcome-closure, the outcome-space $X(A)$ is closed in $\mathbf{E}(A)_+$; by construction, it is also generating. Since $\mathbf{V}(A)_+ \simeq \mathbf{E}(A)_+$, every extremal ray of $\mathbf{E}(A)_+$ consists of multiples of an outcome. Giving $\mathbf{E}(A)$ its standard Jordan structure, primitive idempotents generate extremal rays of $\mathbf{E}(A)_+$, so every primitive idempotent in $\mathbf{E}(A)$ is a positive multiple of an outcome in $X(A)$.

Lemma 8. *Let A be HSD, and let $\mathbf{E}(A)$ have its canonical Jordan structure. Then:*

- (a) *Every extremal unital outcome $x \in X(A)$ is a primitive idempotent.*
- (b) *If A is uniform, then every unital outcome is extremal, hence, a primitive idempotent.*
- (c) *If A is both unital and uniform, it is a Jordan model.*

Proof: (a) Let $x \in X(A)$ be extremal. As discussed above, there then exists some $t > 0$ such that $tx = e$, a primitive idempotent. Now suppose f is a primitive idempotent representing a pure state of \mathbf{E} , with $\langle f, x \rangle = 1$. Then

$$t = t\langle f, x \rangle = \langle f, tx \rangle = \langle f, e \rangle \leq 1,$$

by the Cauchy-Schwarz inequality. Now notice that

$$t^2 \langle x, x \rangle = \langle e, e \rangle = 1$$

so $\langle x, x \rangle = 1/t^2$. Choosing any $E \in \mathcal{M}(A)$ with $x \in E$, we now have

$$1 = \langle e, u \rangle = t\langle x, u \rangle = t \left(\langle x, x \rangle + \sum_{y \in E \setminus \{x\}} \langle x, y \rangle \right) \geq t\langle x, x \rangle = t/t^2 = 1/t,$$

so that $t \geq 1$. Thus, $t = 1$, and $x = e$, a primitive idempotent.

(b) Let $x = \sum_i s_i x_i$ where the x_i are extremal outcomes and $s_i \geq 0$. Let μ be the uniform state on \mathbf{E} . Then

$$\frac{1}{m} = \mu(x) = \sum_i s_i \mu(x_i) = \sum_i s_i \frac{1}{m}$$

so $\sum_i s_i = 1$. If x is unital, therefore, there exists a primitive idempotent f with

$$1 = \langle f, x \rangle = \sum_i s_i \langle f, x_i \rangle.$$

Since the coefficients s_i are convex, we have $\langle f, x_i \rangle = 1$ for every i with $s_i \neq 0$. But then, every x_i is a unital extremal outcome and so, by part (a), a primitive idempotent. It follows (again by the Cauchy-Schwarz inequality) that $s_i \neq 0$ implies $x_i = f$, whence, $x = f$ is again a primitive idempotent. (c) now follows at once from (a) and (b). \square

5.4 Composites of Jordan Models

Suppose a probabilistic theory \mathcal{C} consists entirely of Jordan models. Under what conditions can one equip \mathcal{C} with an associative compositional structure so as to obtain a *monoidal* probabilistic theory? Subject to two further requirements, this is possible *only* if \mathcal{C} is in fact a standard quantum theory:

Theorem 21 ([18]). *Let \mathcal{C} be a symmetric monoidal category of Jordan probabilistic models such that (i) for every $A, B \in \mathcal{C}$, the composite AB is locally tomographic, and (ii) at least one system in \mathcal{C} has the structure of a qubit. Then every model in \mathcal{C} is the hermitian part of a complex matrix algebra.*

The proof of this result exploits the following theorem due to H. Hanche-Olsen.

Theorem 22 (Hanche-Olsen). *If \mathbf{E} is a JC (check) algebra and \mathbf{M}_2 is the Jordan algebra of 2×2 hermitian matrices over \mathbb{C} , then \mathbf{E} is the Hermitian part of a complex matrix algebra iff there exists a Jordan product on $\mathbf{E} \otimes \mathbf{M}_2$ such that*

$$(a \otimes \mathbf{1}) \bullet (b \otimes \mathbf{1}) = ab \otimes \mathbf{1} \text{ and } (\mathbf{1} \otimes x) \bullet (\mathbf{1} \otimes y) = \mathbf{1} \otimes xy \quad (15)$$

for all $a, b \in \mathbf{E}$ and all $x, y \in \mathbf{M}_2$.

Essentially, [18] shows that if AB is a non-signaling HSD composite of HSD models A and B , then local tomography forces the Jordan product on $\mathbf{E}(AB)$ to satisfy (15). A key step is the following observation.

Lemma 9. *Suppose A is a Jordan model. Let AA be a non-signaling composite of A with itself. If AA is Jordan, then the trace form on $\mathbf{E}(AA)$ factors.*

Proof: By definition of a composite, if $x, y \in X(A)$, then $x \otimes y$ is an outcome in $X(AA)$. Since x and y are unital in A , $x \otimes y$ is unital in $X(AA)$. Indeed, the pure product state $\langle x| \otimes \langle y|$ assigns $x \otimes y$ probability 1 (again, by definition of a composite). Hence, by Lemma part (b) of Lemma 8, $x \otimes y$ is a primitive idempotent in $\mathbf{E}(AA)$. But then we also have $\langle x \otimes y | x \otimes y \rangle = 1$, and this is the unique pure state with this property. Hence, $\langle x| \otimes \langle y| = \langle x \otimes y|$, so that

$$\langle x \otimes y | a \otimes b \rangle = \langle x | a \rangle \langle y | b \rangle$$

for all $a, b \in \mathbf{E}(A)$. Since $X(A)$ spans $\mathbf{E}(A)$, the same holds with arbitrary elements of $\mathbf{E}(A)$ in place of x and y , i.e., the inner product factors. \square

Local tomography is a strong constraint on a probabilistic theory. The fact that real and quaternionic quantum mechanics are not locally tomographic should at least slightly temper our willingness to adopt it. A classification of non-locally tomographic non-signaling composites of Jordan models is the subject of on-going work.

6 Conclusion

The framework we have sketched here for a post-classical probability theory has several virtues. It is conceptually conservative, mathematically straightforward, and easily accommodates free mathematical constructions, as well as the introduction of further structure (for example, one can readily topologize the concept of a test space; see [68, 69]). Still, at present, what we have is indeed just the sketch of a framework. Its further development offers many interesting opportunities. We close by mentioning five areas for further work.

Quantum Axiomatics. As long as we restrict our attention to finite-dimensional probabilistic models, it seems that there are many different axiomatic packages — that is, many different clusters of plausible constraints — that locate orthodox QM, or its near environs, within the wild landscape of general post-classical probabilistic theories. In addition to the approach via homogeneity and self-duality, sketched in Section 4, there are various derivations of finite-dimensional QM in the spirit of Hardy’s axioms [39], including work by Rau [55], Dakic and Brukner [26], Masanes and Mueller [48] and Chiribella, D’Ariano and Perinotti [23]. A different approach [35] exploits information geometry. There is also the completeness theorem of Selinger [61] for dagger-compact categories. This is not even to mention the various axiomatic treatments of quantum theory given in the older quantum-logical literature. (This last has been criticized as being too “mathematical”, but much of it becomes significantly simpler when specialized to the finite-dimensional case.) It would be of great interest to know how all of these various axiomatizations (most of which share at least a few assumptions), are related to one another. The mathematical framework developed here seems ideal for this task.

Infinite-Dimensional Models Of even greater interest would be to extend the results of these efforts to infinite-dimensional settings. Individually, infinite-dimensional probabilistic models have been well-studied [27, 29], and tools are available for dealing with composites in this setting, too [67]. However, the line of argument developed in Section 5, depending as it does on the Koecher-Vinberg Theorem, does not generalize easily to the infinite-dimensional setting. Efforts in this direction are just getting underway [refs?], but there is a great deal more work to be done.

Quantum Field Theory Algebraic quantum field theory associates an algebra of observables to each open subset of spacetime. An obvious project would be to consider a probabilistic theory in which each such region is associated with a probabilistic model, subject to the constraint that the model associated with a union of spacelike separated regions be a non-signaling composite of the models associated with the regions individually.

Applications; Post-Quantum Information Theory The notion of a probabilistic model is very broad. It would likely be a fruitful exercise to look for applications outside of quantum information and the foundations of quantum mechanics in which models that are neither classical nor quantum arise. In anticipation of this, it would be very reasonable to further develop the post-classical information theory sketched in [12, 62], especially by investigating in some detail such ideas as *channel capacity* in this setting.

The Measurement Problem. Even though we take measurements and measurement-outcomes as primitives, nothing prevents us from asking whether these can be modeled dynamically *within* the formal framework presented here. Certain versions of the measurement problem can be formulated as theorems in this framework, leading one to wonder whether various strategies for resolving the *quantum* measurement problem — e.g., some version of “many worlds” interpretations, or the apparatus of decoherence — have analogues in the setting of a general probabilistic theory. If so, this would shed some light on *how* these interpretive moves work; if not, then the existence of such an analogue could be regarded as another constraint on a probabilistic theory, taking us closer to orthodox QM. A further discussion of these matters can be found in [71].

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